Covering Points with Lines

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Abstract

Given a set of n points in the plane, is it possible to find k lines that cover all the points in the set? We show that although this problem is NP-hard, it can be solved efficiently for small values of k. In particular, we give a $O(nk \log k + k^{2(k+1)})$ algorithm for this problem, and a generalization to higher dimensions.

1 Introduction

We consider the *point cover problem* in R^d: given a set S of n points in R^d and an integer k, is it possible to find a set of k hyperplanes $H = \{h_1, ..., h_k\}$ such that for every point in S, there is a hyperplane in H incident to it. In a dual setting (see e.g. [4]), this is equivalent to the *hyperplane cover problem*: given an arrangement of n hyperplanes in R^d, can we find a set of k points such that there is at least one point on each hyperplane?

In 1982, Megiddo and Tamir proved that this problem is NP-hard [8] even when $d = 2$, and it was recently shown that the corresponding optimization problem is also APX-hard [7][2]. These facts have been used to prove hardness results for several clustering [1] art gallery [2] and covering problems [5]. The two dimensional problem is also reducible to a particular instance of the *set covering* problem where each set in the given set system intersects with any other set in at most one element. It is shown in [7] that approximating the *minimum set cover with intersection 1* within a factor $o(\log n)$ in random polynomial is not possible unless $NP \subseteq ZTIME(n^{O(\log \log n)})$.

However, Johnson [6] shows that any minimum set cover problem can be approximated within a factor $O(\log n)$ using a greedy algorithm. This is also the best known approximation algorithm for the minimum point cover problem. Approximation algorithms for restricted versions and variants of this problem can be found in [1][5].

In this paper, we study the point cover problem under the lens of fixed parameter tractability [3]. In this setting, we identify some parameters of our

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problem – in this case, k and d – which are likely to be small, and look for polynomial time algorithms when these parameters are considered constants, but where the exponent of the polynomial is independent from the parameters k and d.

In contrast, the point cover problem can be solved by looking at all k-tuple of hyperplanes amongst all the $\binom{n}{d}$ hyperplanes defined by any d points of S. For each of these $O(n^{dk})$ tuples, we can check whether it covers all the points in S in $O(kn)$ time, resulting in a $O(kn^{dk+1})$ time algorithm. For d and k constants, the algorithm is polynomial, but the exponent in n depends on the parameters of the problem. Instead, we will be looking for an algorithm of the form $O(p(n)f(d,k))$ where f is some arbitrary function independent of n, and p(n) is some small polynomial in n. We prove:

Theorem 1 *The point cover problem can be solved in $O(nk^{dk})$ time.*

Theorem 2 *The point cover problem can be solved in $O(n(2k)^{d-1}\log k + dk^{d(k+1)})$ time.*

Details appear in the final version.

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