INTERFERENCE!

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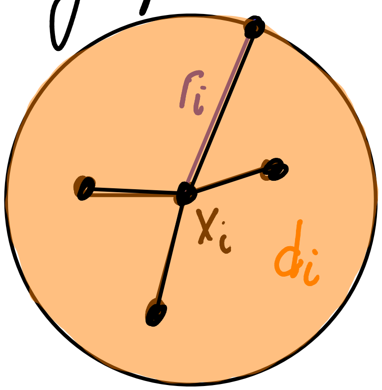
Definitions

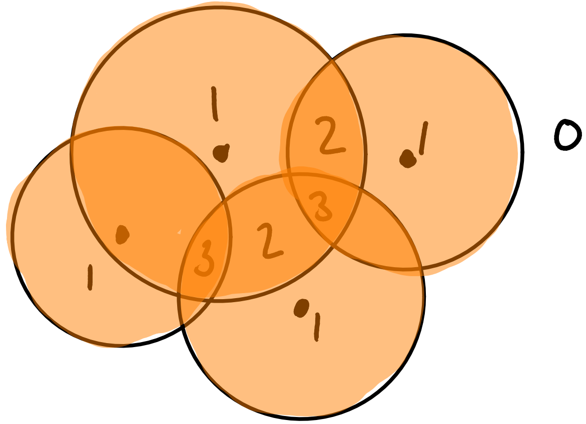
- $G=(V,E)$ is an (undirected) geometric graph
  $V=\{x_1,\dots,x_n\}\subset\mathbb{R}^d$



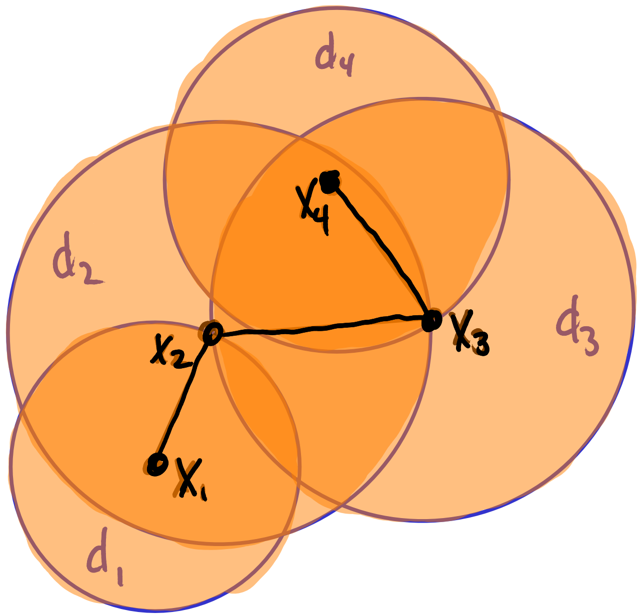
- $r_i = \max\{\|x_i x_j\| : x_i x_j \in E\}$
  $d_i = \{p\in\mathbb{R}^d : \|x_i p\| \le r_i\}$

- $I(p)=|\{i : p\in d_i\}|$

- $I(G)=\max\{I(x_i) : i\in\{1,\dots,n\}\}$



Example



$$I(x_4) = I(G) = 3.$$

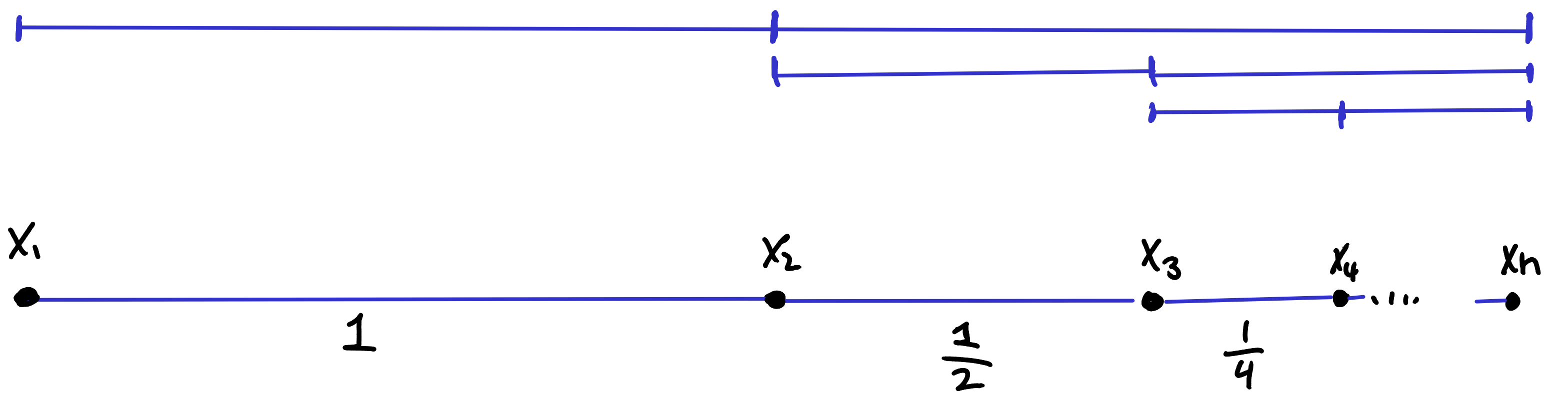
Outline

- Goal: Given $V = \{x_1, \ldots, x_n\} \subset \mathbb{R}^d$, find connected graph $G = (V, E)$ such that $I(G)$ is minimum

- Worst-case
  - There exist V s.t. for any connected G, $I(G) = \Omega(\sqrt{n})$
  - For all V, there exist G such that $I(G) = O(\sqrt{n})$
  - Finding G that minimizes $I(G)$ is NP-hard

- Random point sets.
  - For V i.u.d in $[0,1]$, $I(MST(V)) = \Theta(\sqrt{\log n})$
  - For V i.u.d in $[0,1]^d$, $I(MST(V)) = O(\log n)$
  - For V i.u.d in $[0,1]^d$, there exists G s.t. $I(G) = \Theta(\sqrt{\log n})$

Zeno's Sensor Network



- The obvious solution $E = \{ x_i x_{i+1} : i \in \{1, \dots, n-1\} \}$ has $I(x_n) = n$.

Worst-case lower bound.

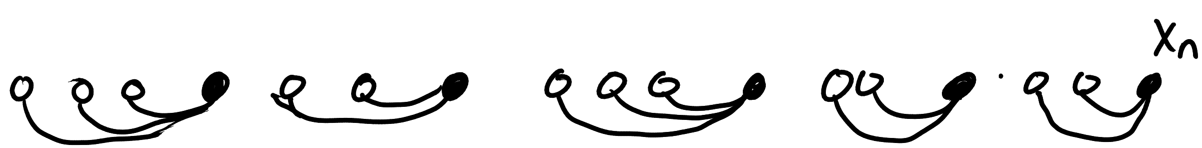
von Rickenbach - Schmid - Wattenhofer - Zollinger 2005

Theorem: Any connected graph on Zeno's sensor network has interference $\Omega(\sqrt{n})$.

Proof: Call x_i a hub if it connects to x_j, $j<i$

- x_n can hear every hub
- every non-hub is connected to a hub

$\Rightarrow$ At least $\sqrt{n}$ hubs, or some hub has degree $\geq \sqrt{n}$ QED.



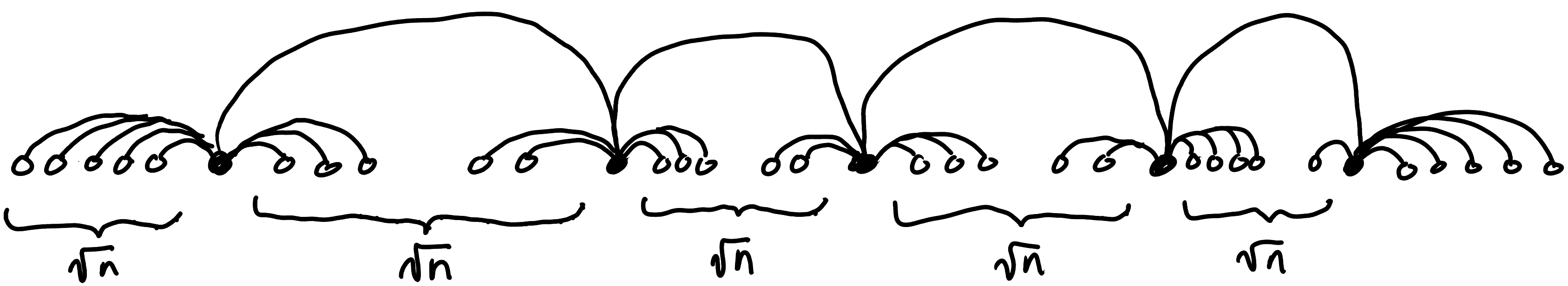
(Zeno's S.N. logarithmic scale)

Worst-case upper bound in $\mathbb{R}$

von Rickenbach - Schmid - Wattenhofer - Zollinger 2005

Theorem: For every $V \subseteq \mathbb{R}$, there exists a connected graph $G=(V,E)$ with $I(G) = O(\sqrt{n})$

Proof:



- $\sqrt{n}$ hubs interfere with everyone
- non-hubs interfere only with members of same group.

Open Problem: Given $V \subset \mathbb{R}$, can we compute $G = (V, E)$ that (approximately) minimizes $I(G)$?

- $O(\sqrt{n})$ upper bound gives an $O(n^{\frac{1}{4}})$-approximation (von Rickenbach - Schmid - Wattenhofer - Zollinger 2005)

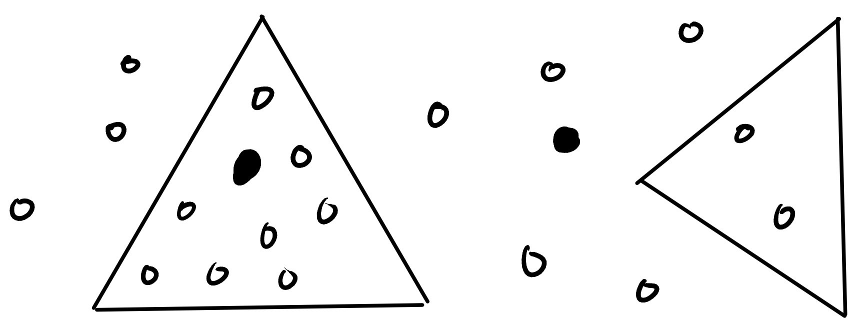
Worst-case upper-bound in $\mathbb{R}^2$

Halldórson-Tokuyama 2008

Theorem: For any $V \subset \mathbb{R}^2$, there exists connected $G=(V,E)$ with
$I(G) = O(\sqrt{n})$.

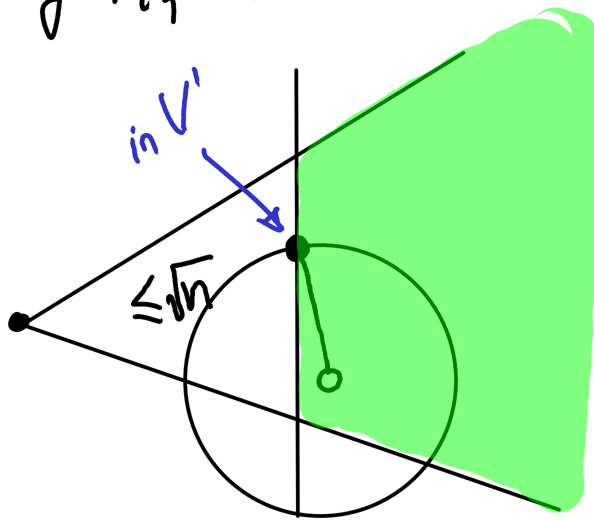
Proof: (ε-nets) - Choose $V' \subset V$, $|V'| = O(\sqrt{n})$, s.t.
for any equilateral Δ, T,

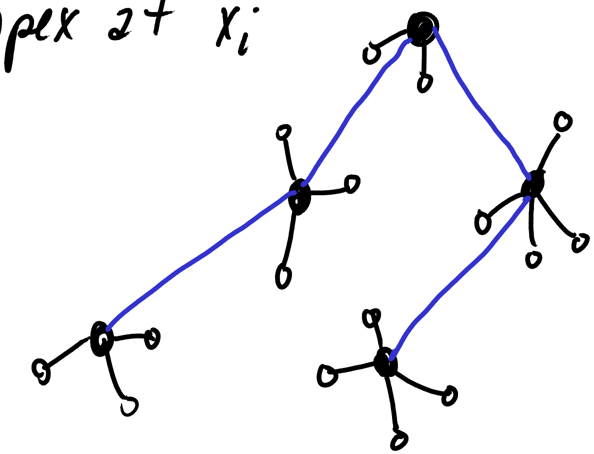
$$|T \cap V| \geq \sqrt{n} \Rightarrow |T \cap V'| > 0$$



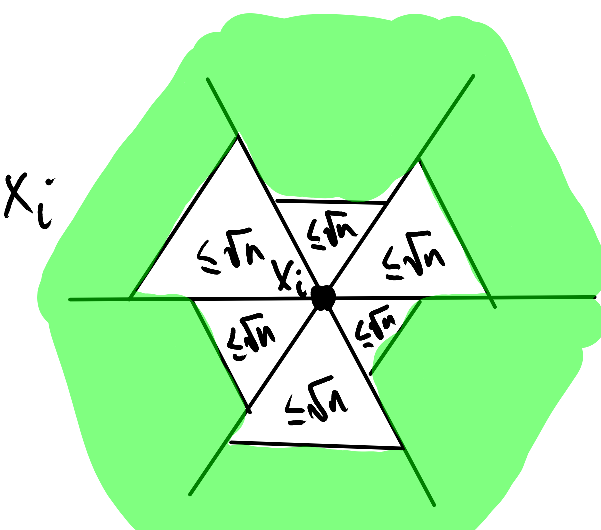
Proof (cont'd).

- Connect V' using any connected graph
- Connect each $x_i \in V \setminus V'$ to nearest element in V'
- For any x_i, consider $60°$ cone with apex at x_i





 doesn't interfere with x_i



Open Problem: Is the following statement true?

For any $V \subset \mathbb{R}^d$, the exists a connected $G=(V,E)$ with $I(G) = O(\sqrt{n})$.

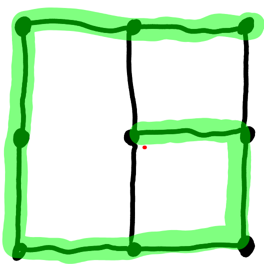
- Previous result extends to $\mathbb{R}^d$, but the bound becomes $O(\sqrt{n \log n})$ [ε-nets are less effient]

NP-Completeness in $\mathbb{R}^2$

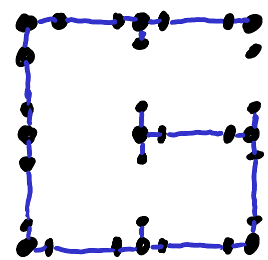
Buchin 2011

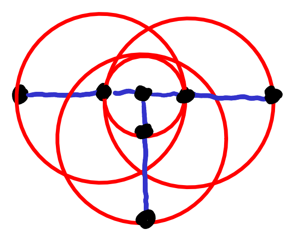
Theorem: For $V \subset \mathbb{R}^2$, testing if there exist a connected $G=(V,E)$ with $I(G) \leq 4$ is NP-hard.

Proof: Reduction from Hamiltonian Path on Max-degree 4 grid graphs.



$\Longrightarrow$





Open Problem: Is there an efficient algorithm for finding a graph G that approximately minimizes $I(G)$?

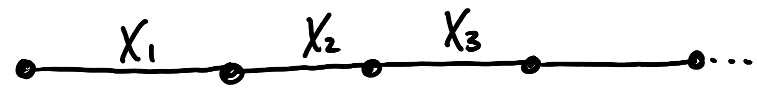
- Buchin's result shows that $5/4$-approximation is best possible

Probabilistic Point Sets – Lower Bound

Kranakis - Krizanc - M - Narayanan - Stacho 2010

Theorem: For V i.u.d in $[0,1]$, $I(MST(V)) = \Omega(\sqrt{n})$ w.h.p.

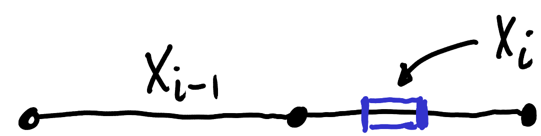
Proof: Look at inter-arrival times



(exponential, $Pr\{X_i \le x\} = 1 - e^{-x} \approx x$, for small x)

k-frame: $X_1, \dots, X_k$ s.t. $1 \le X_1 \le 2$, $\frac{1}{4} X_{i-1} \le X_i \le \frac{1}{2} X_{i-1}$

$Pr\{\text{extending an } i\text{-frame}\} \ge \left(\frac{1}{4}\right)^{i+1}$



$Pr\{X_1 \dots X_k \text{ being a } k\text{-frame}\} \ge \prod_{i=1}^{k} \left(\frac{1}{4}\right)^i = \left(\frac{1}{4}\right)^{\sum i} \approx \left(\frac{1}{4}\right)^{k^2}$ $(*)$

Take $k = \sqrt{\varepsilon \log n}$, $(*) \ge n^{-1/2}$

$\Rightarrow$ We expect to find lots of $\sqrt{\varepsilon \log n}$ - frames.

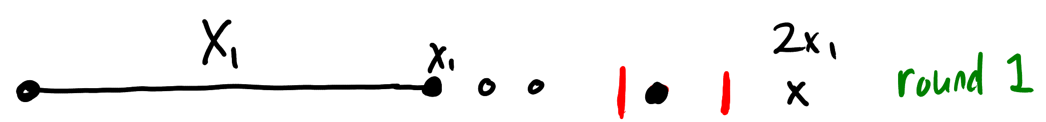
Probabilistic Point Sets — Upper Bound

Kranakis - Krizanc - M - Narayanan - Stacho 2010

Theorem: For V i.u.d in $[0,1]$, $I(MST(V)) = O(\sqrt{n})$ w.h.p.

Proof: · Look at interference coming from left

· Focus on $2x_1$



$$Pr\{\text{round } k \mid \text{round } k-1\} \leq \left(\tfrac{1}{2}\right)^k$$

$$\ldots \; Pr\{K \text{ rounds}\} \leq \left(\tfrac{1}{2}\right)^{K^2} = n^{-c}, \quad \text{for } K = \sqrt{c \log n}$$

Only the last point in each round contributes to $I(2x_1)$.

Open Problem: These results show that, for V i.u.d in $[0,1]$, $I(MST(V)) = \Theta(\sqrt{\log n})$. Is there something better than $MST(V)$, that gives $I(G) \in o(\sqrt{\log n})$?

- The k-frame lower-bound argument only shows that $I(G) \in \Omega((\log n)^{1/4})$.

Probabilistic Point Sets – Upper Bound in $\mathbb{R}^d$

Khabbazian - Durocher - Haghnegahdar 2011

Lemma: For any edge maximal $G=(V,E)$, $I(G) = O\left(\log\left(\frac{e_{max}}{e_{min}}\right)\right)$

longest edge

shortest edge

Theorem: For V i.u.d in $[0,1]^d$, $I(MST(V)) = O(\log n)$ w.h.p.

Proof:

- $e_{max} \leq 1$
- W.h.p $e_{min} \geq 1/n^2$
- Apply Lemma.

Open Problem: What is $I(MST(V))$ when V is i.u.d in $[0,1]^d$?

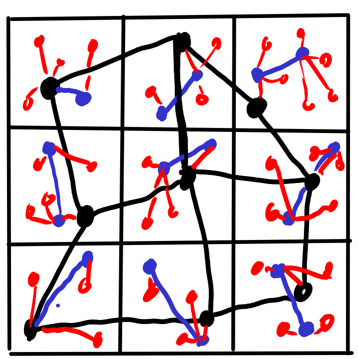
- We only know upper bound of $O(\log n)$.

Probabilistic Point Sets in $\mathbb{R}^d$

Theorem: For V i.u.d in $[0,1]^2$, w.h.p. there exists $G=(V,E)$ such that $I(G) = O(\sqrt{\log n})$

Proof: Partition into grid cells of area $\frac{c \log n}{n}$.

- W.h.p, every cell contains $\Theta(\log n)$ points
- Connect cells into a grid (1 point per cell) [interference $= O(1)$]
- Connect points within each cell using Halldórson-Tokuyama
  - $O(\sqrt{\log n})$ interference within each cell.
- Any cell receives interference from $O(1)$ nearby cells.



Open Problem: Let $G^* = (V, E)$ minimize $I(G^*)$ for fixed V

What is $E[I(G^*)]$ when V i.u.d in $[0,1]^d$

- Previous construction shows upper bound

$$E[I(G^*)] = O(\sqrt{\log n}) \text{ for } d = 2$$

Thank You

for your attention

References

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