COMP2804: Discrete Structures II

Assignment 4

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Administrivia

- Your assignment must be submitted as a single PDF file through cuLearn
- Late assignments will not be accepted under any circumstances. If you're unable to complete the assignment due to a valid and documented medical or personal situation then the weight of this assignment can be shifted to the weight of the remaining assignments.
- You are encouraged to collaborate on assignments, but at the level of discussion only. When writing your solutions, you must do so in your own words.
- Past experience has shown conclusively that those who do not put adequate effort into the assignments do not learn the material and have a probability near 1 of doing poorly on the exams.
- When writing your solutions:
  - You must justify your answers.
  - The answers should be concise, clear and neat.
  - When presenting proofs, every step should be justified.

Meat

ID

1. Make sure the first thing on page 1 of your assignment is your name and student number.

1. Rolling two D20

Consider what hapens when we roll two 20-sided dice d_1 and d_2 (so the sample space is $S = \{(d_1, d_2) : d_1, d_2 \in \{1, 2, 3, \ldots, 20\}\}$ and $Pr(\omega) = 1/|S|$ for each $\omega \in S$). Consider the following events:

- A is the event "$d_1 = 13$"
- B is the event "$d_1 + d_2 = 15$"
- C is the event "$d_1 + d_2 = 21$"

Use the definitions of independence and conditional probability to answer these two questions:

1. Are the events A and B independent?
2. Are the events A and C independent?

2. Randomized Leader Election

A group of $n \geq 3$ people $x_0, \ldots, x_{n-1}$ stand around forming a circle facing inward so that $x_{(i+1) \bmod n}$ is standing to the right of x_i for each $i \in \{0, \ldots, n-1\}$. They play the following game, called "Leader Election" that repeats the following two steps until only one or two people, "The Leaders" remain:

- For each $i \in \{0, \ldots, n-1\}$, person i, tosses a fair coin c_i.
- If $c_i = H$ and $c_{(i-1) \bmod n} = c_{(i+1) \bmod n} = T$ then person i leaves the circle.

The two steps above are called a *round* of the game.

1. What is the maximum number of people who leave the game at the end of the first round?
2. We say that a person playing the game *survives* the first round if they don't leave. For a particular person x_i, what is the probability that x_i survives the first round?
3. For a particular person x_i, what is the probability that Person i survives the first r rounds, for some integer $r < \log_2(n/3)$? What is the expected number of people who survive the first r rounds?

3. Sampling With Replacement

We have a biased coin that, when we toss it, comes up tails (T) with probability $2/n$ and comes up heads (H) with probability $1 - 2/n$. Imagine we toss this coin infinitely many times resulting in an infinite sequence $\pi_1, \pi_2, \ldots, \pi_\infty \in \{H, T\}^\infty$.

1. Let X be the index of the first head in the sequence. That is, $\pi_1 = \pi_2 = \cdots = \pi_{X-1} = T$ and $\pi_X = H$. What is $\mathbf{E}[X]$?
2. Let Y be the index of the first tail in the sequence. That is $\pi_1 = \pi_2 = \cdots = \pi_{X-1} = H$ and $\pi_X = T$. What is $\mathbf{E}[Y]$?

4. Sampling Without Replacement

We have $n-2$ beer bottles $b_1, \ldots, b_{n-2}$ and 2 cider bottles c_1 and c_2. Consider a uniformly random permutation $\pi_1, \ldots, \pi_n$ of these n bottles (so that each of the $n!$ permutations is equally likely).

1. Let X be the index of the first beer bottle in the permutation. That is, $\{\pi_1, \ldots, \pi_{X-1}\} \subseteq \{c_1, c_2\}$ and $\pi_X \in \{b_1, \ldots, b_n\}$. What is $\mathbf{E}[X]$?
2. Let Y be the index of the first cider bottle in the permutation. That is $\{\pi_1, \ldots, \pi_{Y-1}\} \subseteq \{b_1, \ldots, b_n\}$ and $\pi_X \in \{c_1, c_2\}$. What is $\mathbf{E}[Y]$?

5. Doing (much) Better by Taking the Min

Let X be a random variable that takes on the values in the set $\{1, \ldots, n\}$ that satisfies the inequality $\Pr(X \geq i) \leq a/i$ for some value $a > 0$ and all $i \in \{1, \ldots, n\}$.

Recall that (or convince yourself that)

$$\mathbf{E}(X) = \sum_{i=1}^{n} i \Pr(X = i) = \sum_{i=1}^{n} \Pr(X \geq i) \ .$$

1. Given what little you know so far, give the best upper bound you can on $\mathbf{E}(X)$.
2. Let X_1 and X_2 be two independent copies of X and let $Z = \min\{X_1, X_2\}$. What can you say about $\Pr(Z \geq i)$?
3. Give the best upper bound you can on $\mathbf{E}(Z)$.
4. Use Euler's result on the Basel Problem to show that $\mathbf{E}(Z)$ has an upper bound that depends only on a (and not on n).