COMP 2804 — Solutions Assignment 1

Question 1:

- Write your name and student number.

Solution:

- Name: Zlatan Ibrahimović
- Student number: 9

Question 2: Let $n \geq 2$ be an integer.

- Determine the number of strings consisting of n characters, where each character is an element of the set $\{a, b, 0\}$.
- Let S be a set consisting of n elements. Determine the number of ordered pairs (A, B), where $A \subseteq S$, $B \subseteq S$, and $A \cap B = \emptyset$.
- Let S be a set consisting of n elements. Consider ordered pairs (A, B), where $A \subseteq S$, $B \subseteq S$, and $|A \cap B| = 1$. Prove that the number of such pairs is equal to $n \cdot 3^{n-1}$.

Solution: The first part is a standard application of the Product Rule: The procedure is "write a string consisting of n characters, where each character is one of the symbols a, b, and 0". For $i = 1, 2, \ldots, n$, the i-th task is "write one of the symbols a, b, and 0".

For each i, there are 3 ways to do the i-th task; this does not depend on how we did the previous $i - 1$ tasks. By the Product Rule, there are 3^n ways to do the entire procedure. Thus, the answer to this part is 3^n.

For the second part, the trick is to notice that this is the same as the first part: Let $S = \{s_1, s_2, \ldots, s_n\}$. Each ordered pair (A, B), where $A \subseteq S$, $B \subseteq S$, and $A \cap B = \emptyset$, can be specified by a string of n symbols, each symbol being one of a, b, and 0: If $s_i \in A$, then we write the symbol a at position i. If $s_i \in B$, then we write the symbol b at position i. If s_i is not in A and not in B, then we write the symbol 0 at position i. For example, if $n = 5$, $A = \{s_2, s_5\}$, and $B = \{s_1\}$, then we obtain the string $(b, a, 0, 0, a)$.

It should be clear that this gives us a bijection between the objects in the first part and the objects in the second part. As a result, these two parts have the same answer: 3^n.

For the third part, we are going to use the Product Rule:

- The procedure is: Choose an ordered pair (A, B), where $A \subseteq S$, $B \subseteq S$, and $|A \cap B| = 1$.
- First task: Choose an element of S and call it x. (This element will form the intersection of A and B.) There are n ways to do this.

- Second task: Take the set $S' = S \setminus \{x\}$ and choose an ordered pair (A', B'), where $A' \subseteq S'$, $B' \subseteq S'$, and $A' \cap B' = \emptyset$. Since S' has size $n-1$, it follows from the previous part that there are 3^{n-1} ways to do this.

- Third task: Let $A = A' \cup \{x\}$ and $B = B' \cup \{x\}$. There is one way to do this.

By the Product Rule, the total number of ways to do this procedure is equal to $n \cdot 3^{n-1} \cdot 1 = n \cdot 3^{n-1}$.

Question 3: Consider 10 male students $M_1, M_2, \ldots, M_{10}$ and 7 female students $F_1, F_2, \ldots, F_7$. Assume these 17 students are arranged on a horizontal line such that no two female students are standing next to each other. How many such arrangements are there? (The order of the students matters.)
Hint: Use the Product Rule. What is easier to count: Placing the female students first and then the male students, or placing the male students first and then the female students?

Solution: We are going to use the Product Rule:

- The first task is to place the 10 male students on a horizontal line. There are 10! ways to do this.

- After the first task has been completed, consider the following 11 positions: One is to the left of the males, one is to the right of the males, and there is one between any two neighboring males. In each of these 11 positions, we can place at most one female student. In the second task, we choose a subset consisting of 7 of these 11 positions. There are $\binom{11}{7}$ ways to do this.

- In the third task, we place one female in each of the 7 chosen positions. There are 7! ways to do this.

By the Product Rule, the answer to this question is

$$10! \cdot \binom{11}{7} \cdot 7! = 6,035,420,160,000.$$

Question 4: Elisa Kazan[1] has a set $\{C_1, C_2, \ldots, C_{50}\}$ consisting of 50 cider bottles. She divides these bottles among 5 friends, so that each friend receives a subset consisting of 10 bottles. Determine the number of ways in which Elisa can divide the bottles.

Solution: We are going to use the Product Rule. Denote Elisa's friends by $F_1, \ldots, F_5$.

- Task 1: Choose a subset of 10 bottles and give them to F_1. There are $\binom{50}{10}$ ways to do this.

- Task 2: Out of the remaining 40 bottles, choose a subset of size 10 and give them to F_2. There are $\binom{40}{10}$ ways to do this.

[1] President of the Carleton Computer Science Society

- Task 3: Out of the remaining 30 bottles, choose a subset of size 10 and give them to F_3. There are $\binom{30}{10}$ ways to do this.
- Task 4: Out of the remaining 20 bottles, choose a subset of size 10 and give them to F_4. There are $\binom{20}{10}$ ways to do this.
- Task 5: Give the remaining 10 bottles to F_4. There is 1 way to do this.

By the Product Rule, the answer to this question is

$$\binom{50}{10} \cdot \binom{40}{10} \cdot \binom{30}{10} \cdot \binom{20}{10} = 48,334,775,757,901,219,912,115,629,238,400.$$

Question 5: Let $f \geq 2$, $m \geq 2$, and $k \geq 2$ be integers such that $k \leq f$ and $k \leq m$. The Carleton Computer Science program has f female students and m male students. The Carleton Computer Science Society has a Board of Directors consisting of k students. At least one of the board members is female and at least one of the board members is male. Determine the number of ways in which a Board of Directors can be chosen.

Solution: We define the following three sets:

- U: the set of all k-member Board of Directors (without any restrictions).
- A: the set of all k-member Board of Directors consisting of female students only.
- B: the set of all k-member Board of Directors consisting of male students only.

Then we have to determine the size of the set

$$\overline{A} \cap \overline{B},$$

which is the same as the size of the set

$$\overline{A \cup B},$$

which is the same as

$$|U| - |A \cup B|.$$

Since each element of the set U is a subset of k students, we have

$$|U| = \binom{m+f}{k}.$$

Since each element of the set A is a subset of k female students, we have

$$|A| = \binom{f}{k}.$$

Since each element of the set B is a subset of k male students, we have

$$|B| = \binom{m}{k}.$$

Since $A \cap B = \emptyset$, we have

$$|A \cup B| = |A| + |B| - |A \cap B| = \binom{f}{k} + \binom{m}{k} - 0 = \binom{f}{k} + \binom{m}{k}.$$

We conclude that the answer to this question is

$$|U| - |A \cup B| = \binom{m+f}{k} - \binom{f}{k} - \binom{m}{k}.$$

Question 6: You have won the first prize in the *Louis van Gaal Impersonation Contest*[2]. When you arrive at Louis' home to collect your prize, you see n beer bottles $B_1, B_2, \ldots, B_n$, n cider bottles $C_1, C_2, \ldots, C_n$, and n wine bottles $W_1, W_2, \ldots, W_n$. Here, n is an integer with $n \geq 2$. Louis tells you that your prize consists of one beer bottle of your choice, one cider bottle of your choice, and one wine bottle of your choice.

Prove that

$$n^3 = (n-1)^3 + 3(n-1)^2 + 3(n-1) + 1,$$

by counting, in two different ways, the number of ways in which you can choose your prize.

Solution: First way of counting: You can choose your prize by choosing one beer bottle (n choices), one cider bottle (n choices), and one wine bottle (n choices). By the Product Rule, the number of ways to choose your prize is equal to

$$n \cdot n \cdot n = n^3. \tag{1}$$

Second way of counting: Any way of choosing your prize is of exactly one of the following four possible cases:

- Case 1: Do not choose any bottle with index n.

  For this case, there are $n-1$ choices for each bottle. Thus, there are $(n-1)^3$ ways for this case.

- Case 2: Choose exactly one bottle with index n.
  - We choose one of the categories B, C, and W. There are $\binom{3}{1} = 3$ ways to do this.
  - Choose the bottle with index n in the chosen category. There is one way to do this.

[2] Louis van Gaal has been coach of AZ, Ajax, Barcelona, Bayern München, Manchester United, and the Netherlands.

  - For each of the two other categories, choose one of the indices $1, 2, \ldots, n-1$. There are $(n-1)^2$ ways to do this.

  The number of ways to do this second case is equal to

$$3 \cdot 1 \cdot (n-1)^2 = 3(n-1)^2.$$

- Case 3: Choose exactly two bottles with index n.
  - We choose two of the categories B, C, and W. There are $\binom{3}{2} = 3$ ways to do this.
  - For each of the chosen categories, choose the bottle with index n in this category. There is one way to do this.
  - For the remaining category, choose one of the indices $1, 2, \ldots, n-1$. There are $n-1$ ways to do this.

  The number of ways to do this third case is equal to

$$3 \cdot 1 \cdot (n-1) = 3(n-1).$$

- Case 4: Choose three bottles with index n. There is one way to do this.

Since the four cases are pairwise disjoint, the total number of ways to choose your prize is equal to

$$(n-1)^3 + 3(n-1)^2 + 3(n-1) + 1. \tag{2}$$

Since (1) and (2) count the same things, these expressions must be equal.

Question 7: Let $a \geq 0$, $b \geq 0$, and $n \geq 0$ be integers, and consider the set $S = \{1, 2, 3, \ldots, a+b+n+1\}$.

- How many subsets of size $a+b+1$ does S have?
- Let k be an integer with $0 \leq k \leq n$. Consider subsets T of S such that $|T| = a+b+1$ and the $(a+1)$-st smallest element in T is equal to $a+k+1$. How many such subsets T are there?
- Use the above results to prove that

$$\sum_{k=0}^{n} \binom{a+k}{k} \binom{b+n-k}{n-k} = \binom{a+b+n+1}{n}.$$

Solution: The answer to the first part is of course

$$\binom{a+b+n+1}{a+b+1}.$$

For the second part, we fix an integer k with $0 \leq k \leq n$. We divide the set S into three subsets:

$$\begin{aligned}
L &= \{1, 2, \ldots, a+k\}, \\
M &= \{a+k+1\}, \\
R &= \{a+k+2, a+k+3, \ldots, a+b+n+1\}.
\end{aligned}$$

Any subset T of S such that $|T| = a+b+1$ and the $(a+1)$-st smallest element is equal to $a+k+1$ is obtained in the following way:

- Choose an a-element subset from the set L. There are $\binom{a+k}{a}$ ways to do this.
- Choose the element in M. There is one way to do this.
- Choose a b-element subset from the set R. Since R has size $b+n-k$, there are $\binom{b+n-k}{b}$ ways to do this.

By the Product Rule, the answer to this part of the question is

$$\binom{a+k}{a}\binom{b+n-k}{b}.$$

For the third part, we are again going to count the subsets of size $a+b+1$. We divide all these subsets into groups, based on the $(a+1)$-st smallest element in the subset:

- Group G_k: these are all subsets of size $a+b+1$ for which the $(a+1)$-st smallest element is equal to $a+k+1$.
- The possible values for k are $0, 1, 2, \ldots, n$.

Since each subset of size $a+b+1$ is in exactly one of these groups, we have

$$\sum_{k=0}^{n} |G_k| = \binom{a+b+n+1}{a+b+1}.$$

We have seen in the second part that

$$|G_k| = \binom{a+k}{a}\binom{b+n-k}{b}.$$

It follows that

$$\sum_{k=0}^{n} \binom{a+k}{a}\binom{b+n-k}{b} = \binom{a+b+n+1}{a+b+1}.$$

Using $\binom{x}{y} = \binom{x}{x-y}$, we conclude that

$$\sum_{k=0}^{n} \binom{a+k}{k}\binom{b+n-k}{n-k} = \binom{a+b+n+1}{n}.$$

Question 8: In this exercise, we consider strings that can be obtained by reordering the letters of the word ENGINE.

- Determine the number of strings that can be obtained.
- Determine the number of strings in which the two letters E are next to each other.
- Determine the number of strings in which the two letters E are not next to each other and the two letters N are not next to each other.

(You do not get marks if you write out all possible strings. You must use the counting rules that you learned in class.)

Solution: For the first part, we use the approach that we have seen in class: The letter E occurs twice, the letter N occurs twice, the letter G occurs once, and the letter I occurs once. Thus, the number of strings that can be obtained is equal to

$$\binom{6}{2}\binom{4}{2}\binom{2}{1}\binom{1}{1} = 180.$$

For the second part, the two letters E must be next to each other. We imagine that EE is one symbol, say X. Then we are looking for the number of strings that can be obtained by reordering the letters of the word XNGIN. Again, we use the approach that we have seen in class: The letter X occurs once, the letter N occurs twice, the letter G occurs once, and the letter I occurs once. Thus, the number of strings that can be obtained is equal to

$$\binom{5}{1}\binom{4}{2}\binom{2}{1}\binom{1}{1} = 60.$$

For the third part, we define the following three sets:

- U: the set of all strings, without any restrictions, that can be obtained from ENGINE.
- A: the set of all strings in which the two letters E are next to each other.
- B: the set of all strings in which the two letters N are next to each other.

Then we have to determine the size of the set

$$\overline{A} \cap \overline{B},$$

which is the same as the size of the set

$$\overline{A \cup B},$$

which is the same as

$$|U| - |A \cup B|.$$

We have seen in the first part that $|U| = 180$. By the Principle of Inclusion and Exclusion, we have

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

We know from the second part that $|A| = 60$. Of course, the set B has the same size, i.e., $|B| = 60$. To determine the size of the set $A \cap B$, we imagine that EE is one symbol, say X, and NN is one symbol, say Y. Then $|A \cap B|$ is the number of strings that can be obtained by reordering the letters of the word XYGI. This gives $|A \cap B| = 4! = 24$. Thus,

$$|A \cup B| = |A| + |B| - |A \cap B| = 60 + 60 - 24 = 96.$$

We conclude that the answer to this part of the question is

$$|U| - |A \cup B| = 180 - 96 = 84.$$

Question 9: The square in the left figure below is divided into nine cells. In each cell, we write one of the numbers -1, 0, and 1.

Use the Pigeonhole Principle to prove that, among the rows, columns, and main diagonals, there exist two that have the same sum. For example, in the right figure below, both main diagonals have sum 0. (Also, the two topmost rows both have sum 1, whereas the bottom row and the right column both have sum -2.)

0	1	0
1	1	-1
-1	0	-1

Solution: The largest possible row/column/diagonal sum is equal to $1+1+1 = 3$, whereas the smallest possible sum is equal to $-1-1-1 = -3$. It follows that each sum is an element of $\{-3, -2, -1, 0, 1, 2, 3\}$. Note that this set has 7 elements.

How many row/column/diagonal sums are there: There are 3 rows, 3 columns, and 2 main diagonals. Thus, the number of sums is equal to 8.

Let us do this in a more formal way: We make 7 boxes, each one being labeled with one element of the set $\{-3, -2, -1, 0, 1, 2, 3\}$. Each row/column/diagonal is placed in the box whose label is the sum of the three numbers involved. By doing this, we place 8 sums into 7 boxes. By the Pigeonhole Principle, there is a box that receives at least two sums. This means that among the rows, columns, and main diagonals, there exist two that have the same sum.

Question 10: Let $d \geq 1$ be an integer. A point p in $\mathbb{R}^d$ is represented by its d real coordinates as $p = (p_1, p_2, \ldots, p_d)$. The *midpoint* of two points $p = (p_1, p_2, \ldots, p_d)$ and $q = (q_1, q_2, \ldots, q_d)$ is the point

$$\left(\frac{p_1 + q_1}{2}, \frac{p_2 + q_2}{2}, \ldots, \frac{p_d + q_d}{2} \right).$$

Let P be a set of $2^d + 1$ points in $\mathbb{R}^d$, all of which have integer coordinates.

Use the Pigeonhole Principle to prove that this set P contains two distinct elements whose midpoint has integer coordinates.
Hint: The sum of two even integers is even, and the sum of two odd integers is even.

Solution: We are going to use 2^d boxes, each one being labeled by a bitstring of length d. Each point $p = (p_1, p_2, \ldots, p_d)$ in the set P is placed in the box having label

$$(p_1 \bmod 2, p_2 \bmod 2, \ldots, p_d \bmod 2).$$

Since the set P has $2^d + 1$ elements, and we have 2^d boxes, the Pigeonhole Principle implies that there is a box that receives at least two points of P.

Let p and q be two distinct points of P that are in the same box. Then

$$(p_1 \bmod 2, p_2 \bmod 2, \ldots, p_d \bmod 2) = (q_1 \bmod 2, q_2 \bmod 2, \ldots, q_d \bmod 2).$$

Thus, for each $i = 1, 2, \ldots, d$, we have

$$p_i \bmod 2 = q_i \bmod 2.$$

If both $p_i \bmod 2$ and $q_i \bmod 2$ are equal to 0, then both p_i and q_i are even and, therefore, $p_i + q_i$ is even and, therefore, $(p_i + q_i)/2$ is an integer.

If both $p_i \bmod 2$ and $q_i \bmod 2$ are equal to 1, then both p_i and q_i are odd and, therefore, $p_i + q_i$ is even and, therefore, $(p_i + q_i)/2$ is an integer.