

# Carleton University

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Final  
Examination  
Winter 2018

DURATION: 2 HOURS

No. of students: 355

Department Name & Course Number: **Computer Science COMP 2804B**

Course Instructor: Michiel Smid

Authorized memoranda:  
Calculator

**Students MUST count the number of pages in this examination question paper before beginning to write, and report any discrepancy to the proctor. This question paper has 12 pages (not including the cover page).**

**This examination question paper MAY be taken from the examination room.**

**In addition to this question paper, students require:**

an examination booklet: no  
a Scantron sheet: yes

**Instructions:**

1. All questions must be answered on the scantron sheet.
2. Write your name and student number on the scantron sheet.
3. You do not have to hand in this examination paper.
4. Calculators are allowed.

**Marking scheme:** Each of the 25 questions is worth 1 mark.

- $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$ .
- $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ .
- Newton:  $(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$ .
- For  $0 < x < 1$ ,  $\sum_{n=0}^{\infty} x^n = 1/(1 - x)$ .
- Geometric distribution: Assume an experiment has a success probability of  $p$ . We perform the experiment until it is successful for the first time. The expected number of times we perform the experiment is  $1/p$ .
- Expected value:  $\mathbb{E}(X) = \sum k \cdot \Pr(X = k)$ . The sum ranges over all values  $k$  that  $X$  can take.

1. You are given 20 beer bottles  $B_1, B_2, \dots, B_{20}$  and 50 cider bottles  $C_1, C_2, \dots, C_{50}$ . Consider subsets of these 70 bottles, consisting of 30 bottles, exactly 12 of which are beer bottles. How many such subsets are there?

(a)  $\binom{20}{12} \cdot \binom{38}{18}$

(b)  $\binom{20}{12} \cdot \binom{50}{18}$

(c)  $\binom{50}{12} \cdot \binom{20}{18}$

(d)  $\binom{70}{30} \cdot \binom{20}{12}$

2. You are given 20 beer bottles  $B_1, B_2, \dots, B_{20}$  and 50 cider bottles  $C_1, C_2, \dots, C_{50}$ . Consider subsets of these 70 bottles, that contain exactly 12 beer bottles (and any number of cider bottles) or exactly 12 cider bottles (and any number of beer bottles). How many such subsets are there?

(a)  $\binom{20}{12} + \binom{50}{12}$

(b)  $\binom{20}{12} + \binom{50}{12} - \binom{20}{12} \cdot \binom{50}{12}$

(c)  $\binom{20}{12} \cdot 2^{50} + \binom{50}{12} \cdot 2^{20}$

(d)  $\binom{20}{12} \cdot 2^{50} + \binom{50}{12} \cdot 2^{20} - \binom{20}{12} \cdot \binom{50}{12}$

3. You are given 20 beer bottles  $B_1, B_2, \dots, B_{20}$  and 50 cider bottles  $C_1, C_2, \dots, C_{50}$ . Consider subsets of these 70 bottles, that contain at least 3 beer bottles (and any number of cider bottles). How many such subsets are there?

(a)  $2^{70} - 2^{50} - 20 - \binom{20}{2}$

(b)  $2^{70} - 2^{50} - 20 \cdot 2^{50} - \binom{20}{2} \cdot 2^{50}$

(c)  $2^{70} - 2^{50} - 20 \cdot 2^{50}$

(d) None of the above.

4. Consider strings consisting of 40 characters, where each character is an element of  $\{a, b, c, d\}$ . How many such strings contain exactly five  $a$ 's or exactly five  $c$ 's?

(a)  $2 \cdot \binom{40}{5} \cdot 3^{35} - \binom{40}{5} \cdot \binom{35}{5} \cdot 2^{30}$

(b)  $2 \cdot \binom{40}{5} \cdot 3^{35}$

(c)  $\binom{40}{5} + \binom{35}{5} - \binom{40}{5} \cdot \binom{35}{5}$

(d)  $2 \cdot \binom{40}{5} - \binom{40}{5} \cdot \binom{35}{5}$

5. Let  $m \geq 2$  and  $n \geq 2$  be integers. What does

$$\binom{m}{2} + \binom{n}{2} + m \cdot n$$

count?

- (a) The number of ways to choose a subset from a set consisting of  $m + n$  elements.
  - (b) The number of ways to choose an ordered pair of 2 elements from a set consisting of  $m + n$  elements.
  - (c) The number of ways to choose a 2-element subset from a set consisting of  $m + n$  elements.
  - (d) None of the above.
6. Nick (your friendly TA) eats lots of bananas. During a period of 7 days, Nick eats a total of 25 bananas. A banana schedule is a sequence of 7 numbers, whose sum is equal to 25, and whose numbers indicate the number of bananas that Nick eats on each day. Three examples of such schedules are  $(3, 2, 7, 4, 1, 3, 5)$ ,  $(2, 3, 7, 4, 1, 3, 5)$ , and  $(3, 0, 9, 4, 1, 0, 8)$ . How many banana schedules are there?

- (a)  $\binom{31}{6}$
- (b)  $\binom{31}{7}$
- (c)  $\binom{32}{6}$
- (d)  $\binom{32}{7}$

7. A string that is obtained by rearranging the letters of the word

BOOGER

is called *awesome*, if the string does not contain the substring OO. Thus, GEOROB is awesome, whereas GREOOB is not awesome. What is the number of awesome strings?

- (a)  $6 \cdot \binom{5}{2} \cdot 3 \cdot 2$
- (b)  $(6 \cdot \binom{5}{2} \cdot 3 \cdot 2) - 5!$
- (c)  $6! - 5!$
- (d)  $(6 \cdot \binom{5}{2} \cdot 3) - 5!$

8. Consider strings consisting of characters, where each character is an element of  $\{a, b, c, d\}$ . Such a string is called *valid*, if it does not contain  $aa$ , it does not contain  $bb$ , it does not contain  $cc$ , and it does not contain  $dd$ .

For any integer  $n \geq 2$ , what is the number of valid strings of length  $n$ ?

- (a)  $4^n - 4(n - 1)$
- (b)  $4^n - 4n$
- (c)  $4 \cdot 3^n$
- (d)  $4 \cdot 3^{n-1}$

9. Consider bitstrings that do not contain 110. Let  $S_n$  be the number of such strings having length  $n$ . Which of the following is true for any  $n \geq 4$ ?

- (a)  $S_n = S_{n-1} + S_{n-2} + 1$
- (b)  $S_n = S_{n-1} + S_{n-2} + 2^{n-2}$
- (c)  $S_n = S_{n-1} + S_{n-2} + 2^{n-3}$
- (d)  $S_n = S_{n-1} + S_{n-2} + S_{n-3}$

10. Consider the recursive algorithm SUNDAYEVENINGEXAM, which takes as input an integer  $n \geq 1$ :

```

Algorithm SUNDAYEVENINGEXAM( $n$ ):
if  $n = 1$ 
  then print "I don't like Sunday evening exams"
else for  $i = 1$  to  $n$ 
  do print "I don't like Sunday evening exams"
  endfor;
  SUNDAYEVENINGEXAM( $n - 1$ )
endif

```

Let  $P(n)$  be the number of times the line "I don't like Sunday evening exams" is printed when running algorithm SUNDAYEVENINGEXAM( $n$ ). Which of the following is true for all  $n \geq 1$ ?

- (a)  $P(n) = 1 + \frac{n(n-1)}{2}$
- (b)  $P(n) = \frac{n(n-1)}{2}$
- (c)  $P(n) = \frac{n(n+1)}{2}$
- (d)  $P(n) = 1 + \frac{n(n+1)}{2}$

11. You roll a fair die 18 times; the rolls are independent of each other. What is the probability that you roll a 5 exactly three times?

(a)  $\binom{18}{3} \cdot (5/6)^{18}$

(b)  $18 \cdot 5^{15}/6^{18}$

(c)  $\binom{18}{3} \cdot 5^{15}/6^{18}$

(d) None of the above.

12. A string  $s_1s_2 \dots s_n$  is called a *palindrome*, if

$$s_1s_2 \dots s_{n-1}s_n = s_ns_{n-1} \dots s_2s_1,$$

i.e., reading the string from left to right gives the same result as reading the string from right to left.

Let  $n \geq 3$  be an odd integer. You are given a string of length  $n$ , in which each character is a uniformly random element of  $\{a, b, c\}$ . The characters are independent of each other. What is the probability that this bitstring is a palindrome?

(a)  $(1/3)^{(n+1)/2}$

(b)  $(1/3)^{(n-1)/2}$

(c)  $(1/2)^{(n+1)/2}$

(d)  $(1/2)^{(n-1)/2}$

13. You are given a uniformly random bitstring of length five. Define the events

$A$  = “the bitstring contains at most four 1’s”,

$B$  = “the bitstring contains an odd number of 1’s”.

What is  $\Pr(A \mid B)$ ?

(a) 12/16

(b) 13/16

(c) 14/16

(d) 15/16

14. You are given two bitstrings  $a_1, a_2, \dots, a_{77}$  and  $b_1, b_2, \dots, b_{77}$  of length 77. In both bitstrings, each bit is 0 with probability  $3/4$ , and 1 with probability  $1/4$  (independent of all other bits). Consider the string

$$a_1 - b_1, a_2 - b_2, \dots, a_{77} - b_{77}.$$

What is the probability that each element in this string is non-zero?

- (a)  $(3/8)^{77}$
  - (b)  $(4/8)^{77}$
  - (c)  $(5/8)^{77}$
  - (d)  $(6/8)^{77}$
15. You flip a fair coin four times; these four flips are independent. Define the events

$$\begin{aligned} A &= \text{“the first two flips result (in this order) in } HT\text{”}, \\ B &= \text{“the second and third flips result in } TT\text{”}. \end{aligned}$$

Which of the following is correct?

- (a) The events  $A$  and  $B$  are independent.
  - (b) The events  $A$  and  $B$  are not independent.
  - (c) None of the above.
  - (d) All of the above.
16. You flip a fair coin five times; these five flips are independent. Define the events

$$\begin{aligned} A &= \text{“the first three flips result in } HHH\text{”}, \\ B &= \text{“the number of } T \text{ in these five flips is at least two”}. \end{aligned}$$

Which of the following is correct?

- (a) The events  $A$  and  $B$  are independent.
- (b) The events  $A$  and  $B$  are not independent.
- (c) None of the above.
- (d) All of the above.

17. Let  $n \geq 1$  be an integer. Consider a uniformly random permutation of the set  $\{1, 2, 3, \dots, 2n\}$ . Define the event

$A =$  “both the first element and the last element in the permutation are even integers”.

What is  $\Pr(A)$ ?

- (a)  $\frac{2(2n-1)}{n-1}$
  - (b)  $\frac{n-1}{4n}$
  - (c)  $\frac{n-1}{2(2n-1)}$
  - (d)  $\frac{n}{2(2n-1)}$
18. You flip a fair red coin once, and you flip a fair blue coin once, independently of each other. Define the random variables

$$X = \begin{cases} 1 & \text{if the red coin flip resulted in heads,} \\ 0 & \text{if the red coin flip resulted in tails,} \end{cases}$$

$$Y = \begin{cases} 1 & \text{if the blue coin flip resulted in heads,} \\ 0 & \text{if the blue coin flip resulted in tails,} \end{cases}$$

and

$$Z = \min(X, Y).$$

What is the expected value  $\mathbb{E}(Z)$  of the random variable  $Z$ ?

- (a)  $1/4$
  - (b)  $1/2$
  - (c)  $3/4$
  - (d)  $1$
19. Let  $n \geq 2$  be an integer. Consider a bitstring  $b_1, b_2, \dots, b_n$  of length  $n$ , in which each bit  $b_i$  is 0 with probability  $1/2$ , and 1 with probability  $1/2$  (independent of all other bits). Define the random variable  $X$  to be the number of indices  $i$  with  $1 \leq i < n$  for which  $b_i \cdot b_{i+1} = 0$ .

What is the expected value  $\mathbb{E}(X)$  of the random variable  $X$ ?

*Hint:* Use indicator random variables.

- (a)  $n/4$
- (b)  $(n-1)/4$
- (c)  $3n/4$
- (d)  $3(n-1)/4$



20. Let  $n \geq 2$  be an integer. You are given  $n$  beer bottles  $B_1, B_2, \dots, B_n$  and one cider bottle  $C$ . Consider a uniformly random permutation of these  $n + 1$  bottles. The positions in this permutation are numbered as  $1, 2, 3, \dots, n + 1$ . Define the random variable  $X$  to be

$X =$  the position of the leftmost beer bottle.

What is the expected value  $\mathbb{E}(X)$  of the random variable  $X$ ?

- (a) 1
  - (b)  $\frac{n+2}{n+1}$
  - (c)  $\frac{n+1}{n}$
  - (d)  $\frac{n+3}{n+1}$
21. You are given two independent random variables  $X$  and  $Y$ , where

$$\Pr(X = 0) = \Pr(X = 1) = \Pr(Y = 0) = \Pr(Y = 1) = 1/2.$$

Define the random variable  $Z = X \cdot Y$ . Which of the following is correct?

- (a) The random variables  $X$  and  $Z$  are independent.
- (b) The random variables  $X$  and  $Z$  are not independent.
- (c) None of the above.
- (d) All of the above.

22. Alexa and Shelly want to play the game of Monopoly. They use the following recursive algorithm to decide who goes first:

```
Algorithm WHOGOESFIRST( $k$ ):  
//  $k \geq 1$ , the die is fair, and all rolls are independent  
Alexa rolls the die, let  $a$  be the result;  
Shelly rolls the die, let  $s$  be the result;  
if  $a > s$   
then print Alexa goes first;  
    return  $k$   
endif;  
if  $a < s$   
then print Shelly goes first;  
    return  $k$   
endif;  
if  $a = s$   
then WHOGOESFIRST( $k + 1$ )  
endif
```

The ladies run algorithm WHOGOESFIRST(1), i.e., with  $k = 1$ . Define the random variable  $X$  to be the value of the output of this algorithm.

What is the expected value  $\mathbb{E}(X)$  of the random variable  $X$ ?

- (a)  $3/2$
  - (b)  $5/4$
  - (c)  $5/6$
  - (d)  $6/5$
23. Consider the following statement: For any three random variables  $X$ ,  $Y$ , and  $Z$ ,

$$\mathbb{E}(\min(X, Y, Z)) = \min(\mathbb{E}(X), \mathbb{E}(Y), \mathbb{E}(Z)).$$

which of the following is correct?

- (a) The statement is true.
- (b) The statement is false.
- (c) None of the above.
- (d) All of the above.

24. Elisa Kazan has successfully completed her second year as President of the Carleton Computer Science Society. In order to celebrate this, Elisa throws a party. She invites 15 students; thus, the total number of students at the party is equal to 16. Elisa has bought an unlimited amount of drinks: 5 types  $C_1, C_2, C_3, C_4, C_5$  of cider and 3 types  $B_1, B_2, B_3$  of beer. Each of the 16 students gets 3 drinks; each of these drinks is uniformly, and independently, chosen from the 8 types of drinks.

Define the following random variable  $X$ :

$X =$  the number of students who get exactly 2 ciders.

What is the expected value  $\mathbb{E}(X)$  of the random variable  $X$ ?

*Hint:* Use indicator random variables.

- (a)  $2^4 \cdot 3^2 \cdot 5^2/3^8$
  - (b)  $2^4 \cdot 3^2 \cdot 2^5/8^3$
  - (c)  $2^4 \cdot 3^2 \cdot 5^2/8^3$
  - (d)  $3^2 \cdot 5^2/8^3$
25. Are you happy that this is the last question?
- (a) Yes.



