COMP2804B Midterm Exam

Fall 2019

- All questions must be answered on the scantron sheet.
- Write your name and student number on the scantron sheet.
- You do not have to hand in this examination paper.
- Calculators are allowed.

Marking Scheme: Each of the 17 questions is worth 1 mark.

- Binomial coefficients:
$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$
- Newton's Binomial Theorem:
$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$
- Fibonacci numbers:
$$f_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ f_{n-1} + f_{n-2} & \text{if } n \geq 2 \end{cases}$$

1. Carletons Computer Science program has f female students and m male students, where $f \geq 1$ and $f + m \geq 4$. The Carleton Computer Science Society has a Board of Directors consisting of a President and three Vice-Presidents, all of whom are Computer Science students. The President is female and cannot be a Vice-President. In how many ways can a Board of Directors be chosen?

   (a) $\binom{f+m}{3}$

   (b) $f \cdot \binom{f+m-1}{3}$

   (c) $f \cdot \binom{f+m}{3}$

   (d) $(f-1) \cdot \binom{f+m}{3}$

2. Let k and n be integers with $2 \leq k \leq n$ and consider the set $S = \{1, 2, \ldots, n\}$. What is the number of k-element subsets of S that do not contain 1 *and* do not contain 2?

   (a) $\binom{n-1}{k-1}$

   (b) $\binom{n-1}{k}$

   (c) $\binom{n-2}{k-2}$

   (d) $\binom{n-2}{k}$

3. Let k and n be integers with $2 \leq k \leq n$ and consider the set $S = \{1, 2, \ldots, n\}$. What is the number of k-element subsets of S that do not contain 1 *or* do not contain 2?

   (a) $\binom{n-1}{k} + \binom{n-1}{k}$

   (b) $\binom{n-2}{k}$

   (c) $\binom{n}{k} - \binom{n-2}{k-2}$

   (d) $\binom{n}{k} - \binom{n-1}{k-1} - \binom{n-1}{k-1}$

4. For any integer $n \geq 3$, let B_n be the number of bitstrings of length n in which the first three bits are not all equal. Which of the following is true?

   (a) $B_n = 2 \cdot 2^{n-3}$

   (b) $B_n = 6 \cdot 2^{n-3}$

   (c) $B_n = 2^n - 2$

   (d) $B_n = 2^n - 6$

5. Consider strings of length 4 over the alphabet $\{a, b, c, d\}$. How many such strings are there that start with ad or end with dcb?

   (a) 17

   (b) 18

   (c) 19

   (d) 20

6. Let $n \geq 5$ and consider strings of length n over the alphabet $\{a, b, c, d\}$. How many such strings are there that start with ad or end with dcb?

   (a) $4^{n-2} + 4^{n-3} - 4^{n-5}$

   (b) $4^{n-2} + 4^{n-3}$

   (c) $4^n - 4^{n-5}$

   (d) $4^n - 4^{n-2} - 4^{n-3}$

7. What does

$$\binom{w}{3} \cdot \binom{m}{2} + \binom{w}{4} \cdot m + \binom{w}{5}$$

   count?

   (a) The number of ways to choose 5 people out of a group consisting of w women and m men, where at most 3 women can be chosen.

   (b) The number of ways to choose 5 people out of a group consisting of w women and m men, where at most 3 men can be chosen.

   (c) The number of ways to choose 5 people out of a group consisting of w women and m men, where at least 3 women must be chosen.

   (d) The number of ways to choose 5 people out of a group consisting of w women and m men, where at least 3 men must be chosen.

8. Let $n \geq 2$ be an integer and let S be a set of m integers. What is the minimum value of m such that we can guarantee that S contains at least two elements whose difference is divisible by n?

   (a) $n - 1$

   (b) n

   (c) $n + 1$

   (d) $n + 2$

9. What is the coefficient of $x^{24}y^{26}$ in the expansion of $(5x - 7y)^{50}$?

   (a) $-\binom{50}{24} \cdot 5^{26} \cdot 7^{24}$

   (b) $-\binom{50}{26} \cdot 5^{24} \cdot 7^{26}$

   (c) $\binom{50}{24} \cdot 5^{26} \cdot 7^{24}$

   (d) $\binom{50}{26} \cdot 5^{24} \cdot 7^{26}$

10. The function $f : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ is defined by

$$f(n) = \begin{cases} 7 & \text{if } n = 0 \\ \frac{n}{3} \cdot f(n-1) & \text{if } n \geq 1 \end{cases}$$

What is $f(n)$?

(a) $f(n) = 7 \cdot \frac{n!}{3^n}$
(b) $f(n) = 7^n \cdot \frac{n!}{3^n}$
(c) $f(n) = 7 \cdot \frac{(n+1)!}{3^n}$
(d) $f(n) = 7^n \cdot \frac{(n+1)!}{3^n}$

11. For any integer $n \geq 1$, let B_n be the number of bitstrings of length n that do not contain the substring 11 and do not contain the substring 101. Which of the following is true for any $n \geq 4$?

(a) $B_n = B_{n-1} + B_{n-2}$
(b) $B_n = B_{n-1} + B_{n-3}$
(c) $B_n = B_{n-2} + B_{n-3}$
(d) $B_n = B_{n-2} + B_{n-4}$

12. Let $n \geq 1$ be an integer, and let S_n be the number of ways in which n can be written as a sum of 3s and 4s, such that the order in which the 3s and 4s occur in the sum matters. For example, $S_5 = 0$, because 5 cannot be written as a sum of 3s and 4s. We have $S_{11} = 3$, because

$$11 = 3 + 4 + 4 = 4 + 3 + 4 = 4 + 4 + 3 \ .$$

Which of the following is true for any $n \geq 5$?

(a) $S_n = 2 \cdot S_{n-1}$
(b) $S_n = S_{n-1} + S_{n-2}$
(c) $S_n = S_{n-2} + S_{n-3}$
(d) $S_n = S_{n-3} + S_{n-4}$

13. Consider the following recursive algorithm FIB, which takes as input an integer $n \geq 0$:

FIB(n):
  if $n = 0$ or $n = 1$ **then**
    $f = n$
  else
    $f = \text{FIB}(n-1) + \text{FIB}(n-2)$
  return f

When running FIB(12), how many calls are there to FIB(8)?

(a) 4
(b) 5
(c) 6
(d) 7

14. Consider the following recursive algorithm ELISADRINKSCIDER, which takes as input an integer $n \geq 1$, which is a power of 2:

ELISADRINKSCIDER(n):
  if $n = 1$ **then**
    then order Fibonachos
  else
    ELISADRINKSCIDER($n/2$)
    drink n bottles of cider
    ELISADRINKSCIDER($n/2$)

For n a power of 2, let $C(n)$ be the total number of bottles of cider that you drink when running algorithm ELISADRINKSCIDER(n). Which of the following is true for any $n \geq 1$ that is a power of 2?

(a) $C(n) = n \log n - 1$
(b) $C(n) = n \log n + 1$
(c) $C(n) = n \log n$
(d) $C(n) = 2n \log n$

15. You flip a fair coin 9 times. Define the event

$$A = \text{“the result of the first flip is not equal to the result of the second flip”} \ .$$

What is $\Pr(A)$?

(a) 1/4
(b) 1/3
(c) 1/2
(d) 1

16. Consider 4 people, each of which has a uniformly random birthday. We ignore leap years; thus, one year has 365 days. Define the event

$$A = \text{“at least 2 of these 4 people have the same birthday”} \ .$$

What is $\Pr(A)$?

(a) $\binom{4}{2} \cdot \frac{1}{365}$

(b) $\binom{4}{2} \cdot \frac{1}{365} + \binom{4}{3} \cdot \frac{1}{365^2} + \binom{4}{4} \cdot \frac{1}{365^3}$

(c) $1 - \frac{361 \cdot 362 \cdot 363 \cdot 364}{365^4}$

(d) $1 - \frac{362 \cdot 363 \cdot 364}{365^3}$

17. In the game of Hearthstone, you are given a deck of 18 distinct cards. One of the cards is the *Raven Idol*. You choose a uniformly random hand of 3 cards. Define the event

$$A = \text{“the hand of 3 cards contains the Raven Idol”} \ .$$

What is $\Pr(A)$?

(a) $3/17$

(b) $3/18$

(c) $4/19$

(d) $1 - (17/18)^3$

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