COMP2804 Midterm Exam, Winter 2022

First Last 100000000

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This is a closed book exam. You are to do this exam on your own without consulting anyone else or using the internet.

Submit your answers at this URL: https://forms.gle/1mDZw4etNvKst8CK9
(Copy and paste the link into your browser if necessary.)

Marking Scheme: Each of the 17 questions is worth 1 mark.

Reminders:

- Binomial coefficients:
$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$
- Newton's Binomial Theorem:
$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$
- Fibonacci numbers:
$$f_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ f_{n-1} + f_{n-2} & \text{if } n \geq 2 \end{cases}$$

Question 1 (a)**.** The School of Computer Science has f full professors, a associate professors, t assistant professors, i instructors, and s students. The SCS Executive Committee must consist of

- 2 full (f) professors;
- 1 or 2 associate (a) professor;
- 1 assistant (t) professor;
- 1 instructor (i); and
- 2 students (s).

How many ways are there to form an SCS Executive Committee?

✗ $\binom{f}{2} \cdot \binom{a}{2} \cdot t \cdot i \cdot \binom{s}{2}$
✗ $\binom{f}{2} \cdot a \cdot t \cdot i \cdot \binom{s}{2}$
✓ $\binom{f}{2} \cdot \left(a + \binom{a}{2}\right) \cdot t \cdot i \cdot \binom{s}{2}$
✗ $\binom{f}{2} \cdot 2a \cdot t \cdot i \cdot \binom{s}{2}$
✗ $\binom{f}{2} \cdot a^2 \cdot t \cdot i \cdot \binom{s}{2}$

Question 2 (a)**.** A class contains $n \geq 2$ distinct students and wants to send a group of $k \leq n-2$ students on a field trip. Two of these students, Fred and Steve, are bullies. The rest of the students are not bullies.

How many ways are there to choose a set of k students that does not contain any bully?

✗ $\binom{n-1}{k} + \binom{n-2}{k}$
✓ $\binom{n-2}{k}$
✗ $\binom{n-2}{k-1} \cdot (n-k-1)$
✗ $\binom{n-1}{k}$
✗ $\binom{n-2}{k} + 2 \cdot \binom{n-2}{k-1}$

Question 3 (a)**.** Piper (P) and Marley (M) are getting married. In addition to Piper and Marley, the wedding party has p of Piper's friends $P_1, \ldots, P_p$ and m of Marley's friends $M_1, \ldots, M_m$. Piper and Marley have no common friends, so $\{P_1, \ldots, P_p\} \cap \{M_1, \ldots, M_m\} = \emptyset$. This group has to be arranged into a line for a wedding photo.

Marley's friends don't get along with Piper's friends, so we don't want any of Piper's friends to stand next to any of Marley's friends. We also want Piper and Marley to be standing beside each other. For example we could line them up like $P_1, \ldots, P_p, P, M, M_1, \ldots, M_m$.

How many ways are there to line up to the group so that

- none of Piper's friends are standing next to any of Marley's friends and
- Piper and Marley are standing next to each other?

✗ $p!m!$
✗ $2p!m!$
✓ $4p!m!$
✗ $p!m! + p!m!$
✗ $(p+m+1)!(p+m)$

Question 4 (a). Let's go to the animal shelter to take home some cats. At the shelter there are $b \geq 5$ black cats $B_1, \ldots, B_b$ and $w \geq 5$ white cats $W_1, \ldots, W_w$. All the cats are distinct; we can distinguish between any two cats, even if they have the same colour.

How many ways are there to take home 5 cats so that we take home more white cats than black cats?

- ✗ $\binom{b+w}{5}$
- ✗ $\binom{b}{5} + \binom{b}{3} \cdot \binom{w}{2} + \binom{b}{1} \cdot \binom{w}{4}$
- ✗ $\binom{w}{5} + \binom{w}{3} \cdot \binom{b}{2} + \binom{w}{1} \cdot \binom{b}{4}$
- ✓ $\binom{w}{5} + \binom{w}{4} \cdot b + \binom{w}{3} \cdot \binom{b}{2}$
- ✗ $\sum_{k=0}^{5} \binom{w}{k} \cdot \binom{b}{5-k}$

Question 5 (a). Let's go to the animal shelter to take home some cats. At the shelter there are b black cats $B_1, \ldots, B_b$ and w white cats $W_1, \ldots, W_w$. All the cats are distinct; we can distinguish between any two cats, even if they have the same colour.

How many ways are there to take home any number of cats with only requirement being that we take home at least one white cat?

- ✗ 2^{w+b}
- ✗ $2^{w+b} - 1$
- ✓ $2^{w+b} - 2^{b}$
- ✗ $2^{w+b} - 2^{w}$
- ✗ $2^{w+b} - 2^{(w+b)/2}$

Question 6 (a). Let's go to the animal shelter to take home some cats. At the shelter there are b black cats $B_1, \ldots, B_b$ and w white cats $W_1, \ldots, W_w$. All the cats are distinct; we can distinguish between any two cats, even if they have the same colour.

How many ways are there to take home any number of cats with only requirement being that we take home at least one black cat?

- ✗ 2^{w+b}
- ✗ $2^{w+b} - 1$
- ✗ $2^{w+b} - 2^{b}$
- ✓ $2^{w+b} - 2^{w}$
- ✗ $2^{w+b} - 2^{(w+b)/2}$

Question 7 (a). Let $n \geq 6$. How many binary strings of length n begin with 010 or end with 111?

- ✗ 2^{n}
- ✗ 2^{n-6}
- ✓ $2^{n-2} - 2^{n-6}$
- ✗ $2^{n-5} - 2^{n-6}$
- ✗ $2^{n} - 2^{n/2}$

Question 8 (a). How many strings can be obtained by rearranging the letters of the word SUBBOOKKEEPER

- ✗ 13!
- ✓ 13!/48
- ✗ $13 \cdot 12 \cdot \binom{11}{3} \cdot \binom{9}{2} \cdot \binom{7}{2} \cdot \binom{5}{3} \cdot 2 \cdot 1$
- ✗ $13 \cdot 12 \cdot \binom{11}{2} \cdot \binom{9}{3} \cdot \binom{7}{2} \cdot \binom{5}{3} \cdot 2 \cdot 1$
- ✗ $13 \cdot 12 \cdot \binom{11}{2} \cdot \binom{9}{2} \cdot \binom{7}{3} \cdot \binom{5}{3} \cdot 2 \cdot 1$

Question 9 (a). What does $\sum_{k=0}^{m-1} \binom{m}{k}$ count?

- ✓ The number of non-empty subsets of a set of size m
- ✗ The number of empty subsets of a set of size m
- ✗ The number of bitstrings of length m have exactly k many 1s
- ✗ The number of ways to colour the edges of the complete graph K_m with two colours so that there are no monochromatic triangles
- ✗ None of the other answers

Question 10 (a). I have a jar with 33 balls numbered $1, \ldots, 33$. I want to take balls out of the jar until I find two different pairs of balls $\{b_1, b_2\}$ and $\{b_3, b_4\}$ such that $b_1 + b_2 = b_3 + b_4$.

The fewest balls I must take out before I am guaranteed this will happen is:

- ✗ 4 balls
- ✓ 12 balls
- ✗ 15 balls
- ✗ 18 balls
- ✗ 33 balls

Question 11 (a). What is the coefficient of $x^{10}y^{12}$ in the expansion of $(3x - 2y)^{22}$?

- ✗ $\binom{22}{10} \cdot 3^{12} \cdot 2^{10}$
- ✗ $-\binom{22}{10} \cdot 3^{12} \cdot 2^{10}$
- ✓ $\binom{22}{12} \cdot 3^{10} \cdot 2^{12}$
- ✗ $-\binom{22}{12} \cdot 3^{10} \cdot 2^{12}$
- ✗ None of the other answers is correct

Question 12 (a). A string over the alphabet $\{a, b\}$ is called *ab-free* if it does not contain ab. Let $n \geq 2$. How many ab-free strings of length n are there?

- ✗ n
- ✓ $n + 1$
- ✗ f_n
- ✗ f_{n+1}
- ✗ f_{n-1}

Question 13 (a)**.** A bitstring is called 00-free if it does not contain two 0s next to each other. In class we have seen that, for any $m \geq 1$, the number of 00-free bitstrings of length m is equal to the $(m+2)$th Fibonacci number f_{m+2}.

What is the number of 00-free bitstrings of length 30 that have 0 at position 9? (The positions are numbered $1, 2, \ldots, 30$.)

✗ $f_7 \cdot f_{20}$
✗ $f_8 \cdot f_{21}$
✓ $f_9 \cdot f_{22}$
✗ $f_{10} \cdot f_{23}$
✗ None of the other answers is correct

Question 14 (a)**.** A string over the alphabet $\{a, b, c\}$ is called *super* if it does not contain abc, aba, or aa. For $n \geq 1$, let A_n denote the number of super strings of length n. Which of the following is true for any $n \geq 4$?

✗ $A_n = A_{n-1} + A_{n-2} + A_{n-3}$
✓ $A_n = 2A_{n-1} + A_{n-2} + A_{n-3}$
✗ $A_n = 2A_{n-1} + 2A_{n-2} + A_{n-3}$
✗ $A_n = 2A_{n-1} + 2A_{n-2} + 2A_{n-3}$
✗ None of the other answers is correct

Question 15 (a)**.** Consider the following recursive function $\text{Fu}(n)$:

$\text{Fu}(n)$:
  if $n = 0$ **or** $n = 1$ **then**
    return n
  return $\text{Fu}(n-1) + \text{Fu}(n-2)$

When running $\text{Fu}(46)$ how many calls are there to $\text{Fu}(41)$?

✗ 4
✗ 6
✓ 8
✗ 9
✗ 10

Question 16 (a)**.** Let $n \geq 0$ be an integer and let S_n be the number of ways in which n can be written as a sum of 1s, 2s, and 3s, such that the order in which the terms occur matters. For example, $S_3 = 4$ because

$$1+1+1, \;\; 1+2, \;\; 2+1, \;\; 3$$

are all the possible ways of writing 3 as a sum of 1s, 2s, and 3s. Which of the following is true, for any $n \geq 3$?

✗ $S_n = 2S_{n-1}$
✗ $S_n = 4S_{n-3}$
✓ $S_n = S_{n-1} + S_{n-2} + S_{n-3}$
✗ $S_n = S_{n-1} + 2S_{n-2} + S_{n-3}$
✗ $S_n = 3^n$

Question 17 (a)**.** You flip a fair coin 10 times. Define the event:

$$A = \text{“the result of the first and last flip are equal”}$$

What is $\Pr(A)$?

✓ 1/2
✗ 1/4
✗ 1/6
✗ 1/8
✗ 1/16