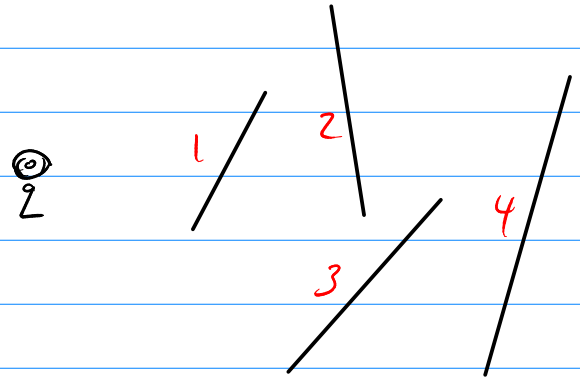
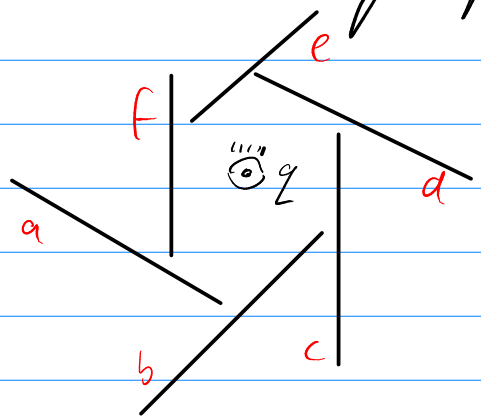
Binary Space Partition Trees

Input: A set S of n line segments in the plane

Problem: Preprocess S so that for any query point q, we can sort S by the "in front of" relationship.



This sorting is not always possible

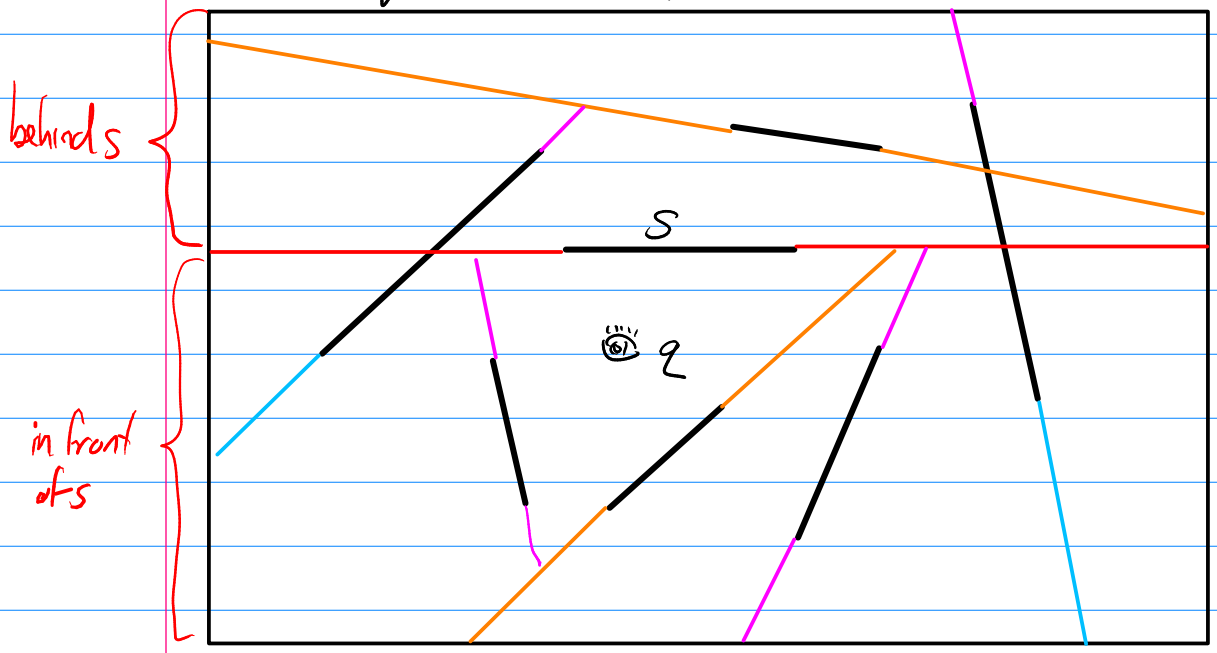


$a < b < c < d < e < f$, where $<$ means "in front of"

- We may have to cut some segments into 2 or more pieces.

BSP tree.

- Pick a random segment $s \in S$
- Extend s into a line, ℓ cutting other segments into 2 as necessary
- Recursively process segments on each side of ℓ

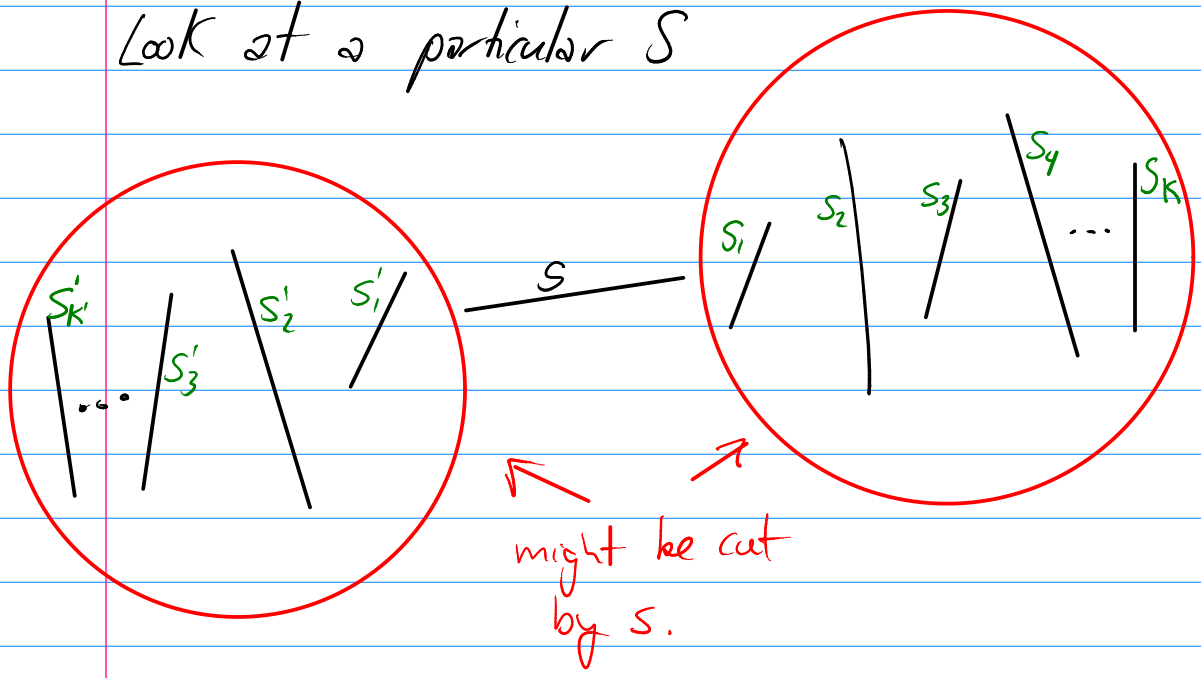


A BSP tree might cut a segment into many pieces

=> The BSP tree could get big!

Size of a BSP tree $= n + \sum_{s \in S} [\#\text{segments cut by } s]$

Look at a particular S



s_i is cut by s if and only if s is the first segment among $\{s, s_1, s_2, \ldots s_i\}$ to be selected.

$\Rightarrow \Pr\{s_i \text{ is cut by } s\} = 1/(i+1)$

Let $I_i = \begin{cases} 1 & \text{if } s_i \text{ is cut by } s \\ 0 & \text{o.w.} \end{cases}$

$I_i' = \begin{cases} 1 & \text{if } s_i' \text{ is cut by } s \\ 0 & \text{o.w.} \end{cases}$

$E[I_i] = E[I_i'] = 0 \cdot \left(1 - \frac{1}{i+1}\right) + 1 \cdot \frac{1}{i+1} = \frac{1}{i+1}$

n_s = # segments cut by s

$$E[n_s] = E\left[\sum_{i=1}^{k} I_i + \sum_{i=1}^{k'} I_i'\right]$$

$$= \sum_{i=1}^{k} E[I_i] + \sum_{i=1}^{k'} E[I_i']$$

$$= \sum_{i=1}^{k} 1/(i+1) + \sum_{i=1}^{k'} 1/(i+1)$$

$$\leq \sum_{i=1}^{n} 1/(i+1) + \sum_{i=1}^{n} 1/(i+1)$$

$$= 2H_n - 2$$

So $E[\text{size of BSP tree}] = E\left[n + \sum_{s \in S} n_s\right]$

$$= n + \sum_{s \in S} E[n_s]$$

$$\leq 2nH_n - n. = O(n \log n).$$

Theorem: For any set S of n disjoint line segments in the plane, a random BSP tree of S has expected size $O(n \log n)$.

Applications

BSP trees are used in ray tracing (movie quality computer graphics), computer-assisted design and manufacturing, and other areas.

BSP trees were instrumental in making the original DOOM run on early 1990's hardware.

This made id Software enormously successful so they could later bring us Wolfenstein 3D, Quake, Rage, Doom 3, Heaven II, Orcs and Elves, ...,