Overhang

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Abstract
How far off the edge of the table can we reach by stacking n identical blocks of length 1? A classical solution achieves an overhang of $\frac{1}{2}H_n$, where $H_n = \sum_{i=1}^{n} \frac{1}{i} \sim \ln n$ is the n^{th} harmonic number, by stacking all the blocks one on top of another with the i^{th} block from the top displaced by $\frac{1}{2i}$ beyond the block below. This solution is widely believed to be optimal. We show that it is exponentially far from optimal by giving explicit constructions with an overhang of $\Omega(n^{1/3})$. We also prove some upper bounds on the overhang that can be achieved. The stability of a given stack of blocks corresponds to the feasibility of a *linear program* and so can be efficiently determined.

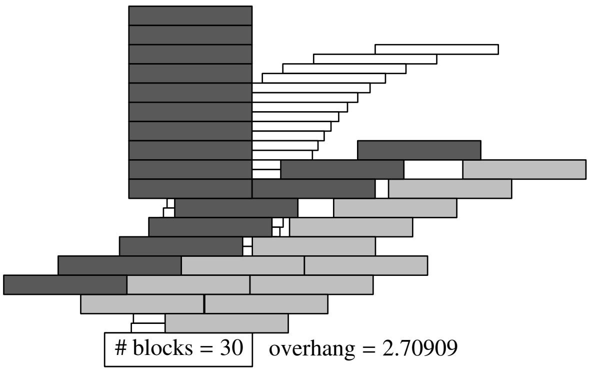

Figure 1: Optimal 30-block stack with overhang 2.709. The "harmonic" 30-block stack with overhang 1.997 is visible behind.

1 Introduction

An attractive problem with a long history is that of stacking some set of objects, such as rectilinear blocks or discs, on a table-top in a stable arrangement with the greatest possible overhang beyond the edge of the table.

In 1923 J. G. Coffin posed the problem in the American Mathematical Monthly "Problems and Solutions" section [C23] but no solution was presented there. The problem recurred from time to time over subsequent decades, e.g., [J55]. The objects to be stacked need not be identical. One of us set a problem of this type, where there were just three uniform thin planks of lengths 2, 3 and 4 to be stacked, for the Archimedeans Problems Drive in 1964 [HP64]. Usually though, the problem is stated in terms of *identical* coins, playing cards, books, bricks, etc. Either deliberately or inadvertently a further restriction often made is that there can be at most one object resting on top of another. Under this restriction the problem has been used by countless teachers as an introduction to recurrence relations, the harmonic series and simple optimisation problems, e.g., [GKP88]. Already for $n = 3$, a larger overhang is reachable without the one-on-one restriction. For $n = 4$, S. Ainley [A79] found the optimum overhang to be $\frac{15-4\sqrt{2}}{8} \sim 1.16789$, a value which was "confirmed by both Bondis". We will show that, for general n, the overhang reachable with a given number of blocks is exponentially larger than that reachable with the restriction.

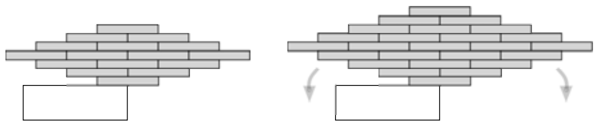

Figure 2: A stable 4-diamond and unstable 5-diamond.

A seductively simple structure, the *m-diamond*, illustrated for $m = 4$ and 5 in Figure 2, consists of a symmetric diamond shape with rows of length $1, 2, \ldots, m-1, m, m-1, \ldots, 2, 1$. The m-diamond uses m^2 blocks and would give an overhang of $m/2$, but unfortunately it is unstable for $m \geqslant 5$. An m-diamond could be stabilised by adding a column of sufficiently many blocks resting on the top block, but the methodology introduced in Section 3 shows that, for $m \geqslant 5$, a column of at least $2^m - m^2 - 1$ blocks would be needed. This solution is however slightly superior to the naive "harmonic tower" solution.

The main results in our paper, Theorems 4.1, 6.1 and 6.2, provide lower and upper bounds for $D(n)$, the maximum overhang that can be constructed using n

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blocks. In Section 2, we give precise definitions for the problems we consider, for $D(n)$ and for related functions. We discuss some natural restrictions we impose and the reasons for these, and introduce the techniques used in deriving our results.

In Section 3 we look at a restricted set of structures, which includes the harmonic towers and diamonds, and prove an upper bound on the overhang. Our empirical results suggest that this bound is fairly tight.

Our best general construction is described in Theorem 4.1 in Section 4, giving an explicit lower bound of $D(n) = \Omega(n^{1/3})$. In Section 5 we give the best empirical lower bounds for $D(n)$ that we have obtained for moderate values of n. Samples of the optimal constructions we have found are illustrated in Figures 1 and 3.

The best upper bounds we have for $D(n)$ (Theorems 6.1 and 6.2) are proved in Section 6. We conjecture that the lower bound of $\Omega(n^{1/3})$ is tight to within a polylogarithmic factor, but our general upper bound falls short of proving this. The paper concludes with some remarks and open problems in Section 7.

2 Preliminaries

As the maximum overhang problem is physical in nature, our first task is to formulate it mathematically. We will only consider a 2-dimensional version of the problem. Each block is a rectangle of length 1 and height h with uniform density and unit weight. (We will see that the height h is unimportant.) We assume that the table occupies the quadrant $x, y \leqslant 0$ of the 2-dimensional plane. A stack is specified by giving the position of each of its blocks. Throughout most of the paper we consider *orthogonal* stacks in which the sides of the blocks are parallel to the axes, with the length of each block parallel to the x axis. The position of a block can then be specified by giving the coordinate (x, y) of its lower left corner. Such a block then occupies the box $[x, x+1] \times [y, y+h]$. A stack composed of n blocks is specified by a sequence $(x_1, y_1), \ldots, (x_n, y_n)$ of the lower left coordinates of its blocks. We require each y_i to be a non-negative integral multiple of h, the height of the blocks. Blocks are not allowed to overlap. The *overhang* of the stack is $1 + \max_{i=1}^{n} x_i$.

A block at position (x_1, y_1) *rests* on a block in position (x_2, y_2) if $|x_1 - x_2| \leqslant 1$ and $y_1 - y_2 = h$. The *interval of contact* between the two blocks is then $[\max\{x_1, x_2\}, 1 + \min\{x_1, x_2\}] \times \{y_1\}$. A block placed at position $(x, 0)$ *rests on the table* if $x \leqslant 0$. When one block rests on other, each may exert various *forces* on the other along their interval of contact. A force is a vector acting at a specified point. By Newton's second law, forces come in opposing pairs. If a force f is exerted on block A by block B at (x, y), then a force $-f$ is exerted on block B by block A, again at (x, y). In general, two blocks may exert a (possibly infinitesimal) force on each other at each point along their interval of contact. However, in our case, it is always possible to replace this collection of forces by a single *resultant* force acting at a single point within their interval of contact.

A force $f = (f_x, f_y)$ may be resolved into a horizontal force of f_x units acting along the x direction and a vertical force of f_y units acting along the y direction. Suppose that block A is resting on block B with (x, y) being a point in their contact interval. Suppose that a force $f = (f_x, f_y)$ is exerted on A by B. As there is nothing that holds the blocks A and B together, the blocks can push, but not pull, one another, i.e., $f_y \geqslant 0$. Furthermore, if the edges of A and B are completely smooth so that there is no *friction* between them, then we also have $f_x = 0$. Throughout the paper, with brief exceptions, we consider *frictionless* blocks, so that all the forces acting between blocks are vertical.

Each block is also subjected to a downward *gravitational force* of one unit acting at its center of mass. As the blocks are assumed to be of uniform density, the center of mass of a block whose lower left corner is at (x, y) is at $(x + \frac{1}{2}, y + \frac{h}{2})$.

A configuration of forces acting between the blocks of a stack, and between the blocks and the table, is *admissible* if it includes the gravitational forces acting on the blocks and if all the non-gravitational forces satisfy all the requirements mentioned above.

A rigid body is said to be in *equilibrium* if the sum of the forces acting on it, and the sum of the moments they apply on it, are both zero. A 2-dimensional rigid body acted upon by k vertical forces $f_1, f_2, \ldots, f_k$ at $(x_1, y_1), \ldots, (x_k, y_k)$ is in equilibrium if and only if $\sum_{i=1}^{k} f_i = 0$ and $\sum_{i=1}^{k} x_i f_i = 0$.

Definition 1. (Stability) *A stack of blocks is* stable *if and only if there is an admissible configuration of forces acting on them under which each block is in equilibrium.*

Static stability problems of the kind considered here are often *under-determined*, so that a stabilizing set of forces, if it exists, is usually not unique. It was actually the consideration by one of us of such stability concerns arising in the game of *Jenga* [Z02] which stimulated this current work. The following theorem shows that the stability of a given stack can be efficiently checked.

Theorem 2.1. *The stability of a stack containing n blocks can be decided by checking the feasibility of a linear program with $O(n)$ variables and constraints.*

Due to lack of space, the (simple) proof is omitted.

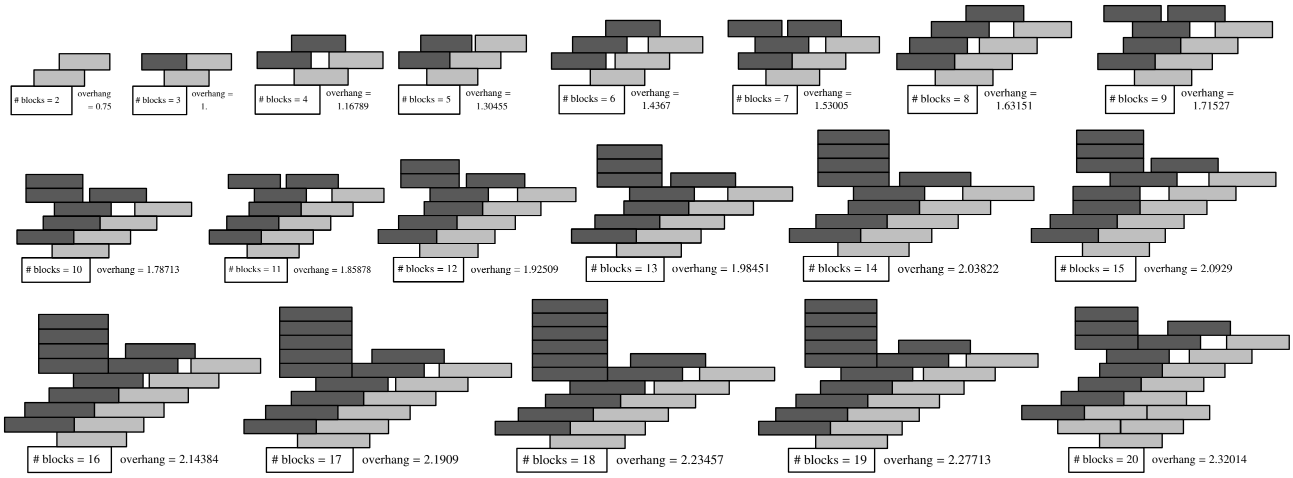

Figure 3: Optimal constructions with up to 20 blocks. The lighter-shaded blocks form the support set while the darker ones form the balancing set. All these constructions are spinal except the last one.

Definitions 2. ($D(n)$ and principal block) *Define $D(n)$ to be the maximum overhang achievable by a stable stack comprising n blocks of length 1. The block of the stack that achieves the maximum overhang is the* principal block *of the stack. If several blocks achieve the maximum overhang, the lowest one is chosen.*

Definitions 3. (Support set, balancing set, D_k) *The* support set *of a stack is defined recursively as follows: the principal block is in the support set, and if a block is in the support set then any block on which this block rests is also in the support set. The* balancing set *consists of any blocks that do not belong to the support set. Define $D_k(n)$ to be the maximum overhang achievable using a total of n blocks of which exactly k are in the support set. Then $D(n) = \max_{1 \leqslant k \leqslant n} D_k(n)$.*

Definition 4. (Loaded stacks, $D_k^*(w)$ and $D^*(w)$) *It is natural and convenient to consider* loaded *stacks, which consist only of a support set with some* point weights *attached to its blocks. Let $D_k^*(w)$ be the maximum overhang achievable using a support set of k blocks with attached point weights of total weight $w - k$, and let $D^*(w) = \max_{1 \leqslant k \leqslant w} D_k^*(w)$.*

Theorem 2.2. *$D_k(n) \leqslant D_k^*(n)$.*

Proof. Consider the set of forces exerted on the support set of a stack by the set of balancing blocks. From the definition of the support set, no block of the support set can rest on any balancing block, therefore the effect of the support set can be represented by a set of *downward* vertical forces on the support set, or equivalently by a finite set of point weights attached to the support set with the same total weight as the set of balancing blocks. □

We have found in our empirical constructions that optimal loaded stacks can usually be either exactly represented or closely approximated by (non-loaded) stacks of the same total weight. We therefore conjecture:

Conjecture 1. *$D(n) = D^*(n) - O(1)$.*

3 Spinal stacks

In this section we focus on a restricted class of stacks for which the analysis is simpler.

Definitions 5. (Spinal stacks, spine) *A stack is* spinal *if its support set (its* spine*) has just a single block at each level.*

In particular, any stack in which the x-coordinates of the rightmost block are increasing in successive rows from the table up to the principal block (a *monotone stack*) is clearly spinal.

Definitions 6. ($S_k(n)$, $S(n)$, $S_k^*(w)$ and $S^*(w)$) *Let $S_k(n)$ and $S(n)$ (corresponding to $D_k(n)$ and $D(n)$) be the maximum overhangs achievable using spinal stacks composed of n blocks, respectively with and without the condition of using a support set of size k. Similarly, we have $S_k^*(w)$ and $S^*(w)$ corresponding to loaded spinal stacks with total weight w.*

It is tempting to make the (false) assumption that spinal stacks are optimal. Indeed we have the following:

Theorem 3.1. *For $1 \leqslant n \leqslant 19$, $D(n) = S^*(n)$.*

Proof. Verified empirically by exhaustive search. □

However the next theorem allows us to show that no 20-block spinal stack, loaded or non-loaded, can reach the overhang illustrated in the last stack of Figure 3.

Notation. Given a loaded spinal stack with k blocks in the spine, we denote the blocks from bottom to top as $B_1, B_2, \ldots, B_k$, so B_k is the principal block, and we may regard the tabletop as B_0. For $1 \leqslant i \leqslant k$, the weight attached to B_i is denoted by w_i and the relative overhang of B_i beyond B_{i-1} is denoted by d_i. We define $t_i = \sum_{r=i}^{k}(1+w_r)$, the total weight exerted upon B_{i-1} from block B_i, and take t_{k+1} to be 0.

THEOREM 3.2. *A loaded spinal stack with total weight w and with k blocks in the spine achieving the maximal overhang of $S_k^*(w)$ satisfies the following conditions:*

1. $d_i = 1 - \frac{t_{i+1}+\frac{1}{2}}{t_i}$ *and so the stack is monotone, i.e., $d_i \geqslant 0$ for $1 \leqslant i \leqslant k$;*
2. *each block is balanced over the righthand edge of the block below it and each weight is attached at the lefthand edge of its block;*
3. *for some j, $0 \leqslant j \leqslant k$, we have $(t_{i+1}+\frac{1}{2})t_{i-1} = t_i^2$ for $j \leqslant i \leqslant k$ and $w_i = 0$ for $1 \leqslant i \leqslant j$.*

Proof. Due to lack of space we omit the proofs of Conditions 1 and 2. It remains to prove Condition 3, and we assume that Conditions 1 and 2 hold. Figure 4 shows a portion of such a loaded spinal stack.

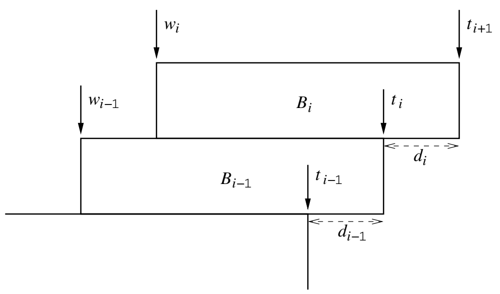

Figure 4: Fragment of loaded spinal stack.

We are now ready to analyse the balance conditions which optimise the total overhang. For any fixed i, $2 \leqslant i \leqslant k$, we take $w_i + w_{i-1} = q$ to be fixed and optimize the contribution of $d_i + d_{i-1}$ to the overhang as $y = w_{i-1}$ varies. Then

$$\begin{aligned} t_i &= t_{i+1} + 1 + w_i, \text{ and} \\ t_{i-1} &= t_i + 1 + w_{i-1}, \end{aligned}$$

so

$$\begin{aligned} t_{i-1} &= t_{i+1} + 2 + q \text{ (fixed), and} \\ t_i(y) &= t_{i+1} + 1 + q - y = t_{i-1} - 1 - y. \end{aligned}$$

By resolving moments about the right end of each block, we find:

$$d_i(y) = \frac{w_i + \frac{1}{2}}{t_i} = \frac{t_i(y) - t_{i+1} - \frac{1}{2}}{t_i(y)} = 1 - \frac{t_{i+1}+\frac{1}{2}}{t_i(y)};$$

$$d_{i-1}(y) = \frac{w_{i-1}+\frac{1}{2}}{t_{i-1}} = \frac{y+\frac{1}{2}}{t_{i-1}}.$$

Differentiating $d_i + d_{i-1}$ with respect to y, we have

$$\frac{d}{dy}(d_i + d_{i-1}) = -\frac{t_{i+1}+\frac{1}{2}}{t_i(y)^2} + \frac{1}{t_{i-1}},$$

which is a decreasing function of y over the range $0 \leqslant y \leqslant q$, and takes a negative value at the endpoint $y = q$. Thus the contribution of $d_i + d_{i-1}$ is maximized with respect to y either where the derivative is zero, i.e., $(t_{i+1}+1/2)t_{i-1} = t_i^2$ if this corresponds to a value of y satisfying $0 \leqslant y \leqslant q$, or else at endpoint $w_{i-1} = y = 0$. Note however that if $w_{i-1} = 0$, then the corresponding analysis for $w_{i-2} + w_{i-1}$ shows that we must also have $w_{i-2} = 0$, and similarly for all other w's down to w_1. This establishes Condition 3. □

THEOREM 3.3. $S^*(w) < \ln w + 1$.

Proof. For fixed total weight $w = t_1$ and fixed k, the largest possible overhang $S_k^*(w) = \sum_{i=1}^{k} d_i$ is attained when the conditions given in Theorem 3.2 all hold. Then

$$\sum_{i=1}^{k} d_i = d_k + \sum_{i=1}^{k-1}\left(1 - \frac{t_{i+1}+\frac{1}{2}}{t_i}\right) < d_k + k - 1 - \sum_{i=1}^{k-1}\frac{t_{i+1}}{t_i}.$$

But $d_k \leq 1$, and, putting $x_i = t_{i+1}/t_i$, we see that

$$S_k^*(w) < k - \sum_{i=1}^{k-1} x_i \quad \text{and} \quad \prod_{i=1}^{k-1} x_i = \frac{t_k}{t_1} \geqslant \frac{1}{w}.$$

The minimum sum for a finite set of positive real numbers with fixed product is attained when the numbers are equal, hence

$$S_k^*(w) < k - (k-1)w^{-\frac{1}{k-1}}.$$

Let z be defined by $k - 1 = z \ln w$, then this becomes

$$S_k^*(w) < 1 + z \ln w(1 - e^{-1/z}) < 1 + \ln w. \qquad \square$$

COROLLARY 3.1. $S(n) < \ln n + 1$.

To justify the claim made in the Introduction concerning the instability of diamond stacks, consider the spine of an m-diamond. In this case, $d_i = 1/2$ for all i and so the balance conditions give the equations

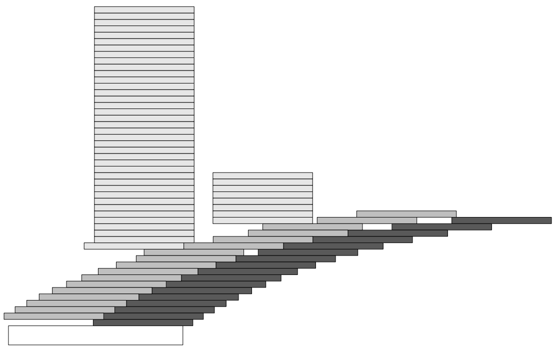

Figure 5: Spinal stack of 100 blocks.

$t_i \geqslant 2t_{i+1}+1$ for $1 \leqslant i \leqslant m-1$. But $t_m \geqslant 1$, so we have $t_i \geqslant 2^i - 1$ for all i and hence $t_1 \geqslant 2^m - 1$. Since t_1 is the total weight of the stack, the number of extra blocks required to be added for stability is at least $2^m - 1 - m^2$, which is positive for $m \geqslant 5$.

We can now describe a construction of loaded spinal stacks which achieve an overhang agreeing asymptotically with the upper bound proved in Theorem 3.3.

Theorem 3.4. *$S^*(w) > \ln w - \ln\ln w + O(1)$.*

Proof. It is convenient to choose all w_i's equal, a good enough value for k, and d_i's to maintain the conditions that each spinal block is only just balanced and has the force w_i applied at its left end.

Let k be the number of blocks in the spine, and let $w_i = v$ for $1 \leqslant i \leqslant k$, where $k + kv = w$. Since $t_{k+1} = 0$ and $t_i = t_{i+1} + 1 + v$, we have

$$t_i = (1+v)(k+1-i), \text{ and so } d_i = \frac{v+\frac{1}{2}}{t_i} = \frac{v+\frac{1}{2}}{v+1} \cdot \frac{1}{k+1-i}$$

for $1 \leqslant i \leqslant k$. Hence

$$S^*(w) \geqslant D = \sum_{i=1}^{k} d_i = \frac{v+\frac{1}{2}}{v+1} \cdot H(k) = \left(1 - \tfrac{1}{2v+2}\right) H(k).$$

With equal w_i's, an approximately optimal choice for k is $k = \lceil 2w/\ln w \rceil$, then

$$\begin{aligned} D &= (1 - \frac{1}{\ln w} + O(\tfrac{1}{w}))(\ln w - \ln\ln w + O(1)) \\ &> \ln w - \ln\ln w + O(1). \end{aligned}$$ □

In Figure 5 we give an example from the optimal spinal stacks that we have constructed. The spine is darkly shaded. Large stabilising forces are supplied by several "towers" (two in this figure, lightly shaded). The role of the "shadow" of the spine (medium shading) is to spread the concentrated loads of the towers onto the tiny steps of the spine. The placement of the support set is unique, but there is a lot of freedom in the placement of the blocks from the balancing set.

The overhang achieved here is about 3.6979, which is a considerable improvement on the 2.5937 given by a harmonic stack, but is also substantially less than the optimal 4.2080 attainable with a non-spinal stack.

We believe that, with few exceptions, the effect of an optimal arrangement of point weights on the spine can be given by an appropriately arranged balancing set.

Conjecture 2. *$S(n) = S^*(n)$ for $n \neq 3, 5,$ or 7.*

We have verified this conjecture numerically for $n \leqslant 1000$.

For $n = 3$, $S(3) = S_2(3) = 1 < \frac{11-2\sqrt{6}}{6} = S_2^*(3) = S^*(3)$. For equality we would need some frictional force and a stack as represented approximately in Figure 5 in order to supply the "optimal" point weights of $w_1 = 2 - \sqrt{3/2}$ and $w_2 = -1 + \sqrt{3/2}$. Note that with *any positive* coefficient of friction, a configuration of this form would be stable for sufficiently thin blocks. At the other extreme, we can obtain an overhang greater than 1 (indeed arbitrarily close to $\sqrt{2}(\sqrt{3}-1)$) with only *two* blocks if the coefficient of friction is sufficiently large and the block height $h \leq 1$ is chosen suitably!

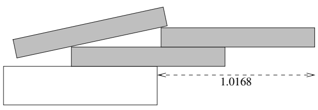

Figure 6: With 3 blocks we would like some friction!

4 Brick wall stacks

As we have indicated, for large enough values of n spinal stacks are not optimal. For example, at $n = 20$ the optimal spinal stack has an overhang of about 2.31899 whereas the non-spinal stack (illustrated in Figure 3 gives the slightly larger overhang of about 2.32014. (Note that there is a scarcely visible gap between the two blocks at the second level.)

We give now a general construction for a sequence of stacks which establishes that $D(n) = \Omega(n^{1/3})$. Although this is not strictly optimal (see the empirical results in Section 5), it gives the best explicit general bound we know and we conjecture that it is within a polylogarithmic factor of optimality. For simplicity, the construction is what we term a *brick-wall* stack, which resembles the simple "stretcher-bond" pattern in real-life bricklaying. In each row the blocks are contiguous, and each is centred over the ends of blocks in the row beneath.

Theorem 4.1. *$D(n) \geqslant (3n/16)^{1/3} - O(1)$ for all n.*

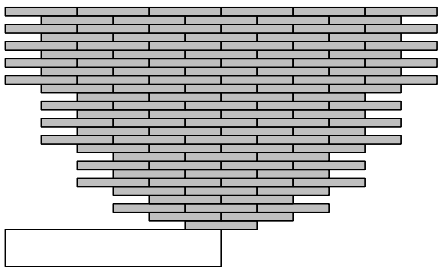

Figure 7: A 5-stack consisting of 111 blocks and giving an overhang of 3.

An illustration of the construction with overhang 3 and 111 blocks is given in Figure 7.

Overall the stack has a symmetric roughly parabolic shape, with vertical axis at the table edge and a brick-wall structure.

A *t-row* is a row of t adjacent blocks, symmetrically placed with respect to $x = 0$. An *r-slab* has height $2r-1$ and consists of alternating $(r+1)$-rows, and r-rows, starting and finishing with $(r+1)$-rows. An r-slab therefore contains $r(r+1)+(r-1)r = 2r^2$ blocks. An r-slab for $r = 5$ is illustrated in Figure 8. A *d-stack* is a d-slab on a $(d-1)$-slab on ... on a 2-slab on a 1-slab on a single block. A 5-stack is illustrated in Figure 7. Our whole construction is just a d-stack and so has overhang $(d+1)/2$, and total number of blocks given by $n = 1+\sum_1^d(2r^2) = 2d^3/3+O(d^2)$. Lemma 4.2 shows that the construction is stable.

In preparation for Lemma 4.2, we show in the next lemma that a slab can concentrate a set of forces acting on its top together with the weights of its own blocks down into a narrower set of forces acting on the row below it. The lemma is illustrated in Figure 8.

LEMMA 4.1. *For any $g \geqslant 0$, an r-slab with forces of $rg, 2rg, 2rg, \ldots, 2rg, rg$ acting downwards onto its top row at positions $-\frac{r+1}{2}, -\frac{r-1}{2}, -\frac{r-3}{2}, \ldots, \frac{r-1}{2}, \frac{r+1}{2}$ respectively, and with forces of $(r+1)g+r, 2(r+1)g+2r, \ldots, 2(r+1)g+2r, (r+1)g+r$ acting upwards on its bottom row at positions $-\frac{r}{2}, -\frac{r-2}{2}, \ldots, \frac{r-2}{2}, \frac{r}{2}$ respectively, is in equilibrium.*

Proof. The proof is by induction on r. For $r = 1$, a 1-slab is just a 2-row, which is clearly stable with downward forces of $g, 2g, g$ at $-1, 0, 1$ and upward forces of $2g+1, 2g+1$ at $-\frac{1}{2}, \frac{1}{2}$.

For the induction step, we first observe that for any $r > 1$ an r-slab can be regarded as an $(r-1)$-slab with an $(r+1)$-row added above and below and with an extra block added at each end of the $r-2$ $(r-1)$-rows of the $(r-1)$-slab. The 4-slab (shaded) contained in a 5-slab together with the added blocks is shown in Figure 8.

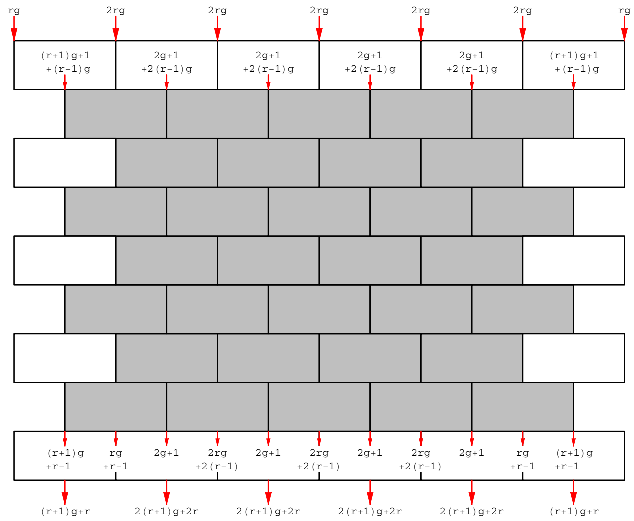

Figure 8: A 5-slab with a grey 4-slab contained in it.

Suppose the statement of the lemma holds for $(r-1)$-slabs and consider an r-slab with the supposed forces acting on its top row. The top row can be balanced by $r+1$ equal forces of $2rg+1$ (the 1 is for the block in the top row) acting at positions $-\frac{r}{2}, -\frac{r-2}{2}, \ldots, \frac{r-2}{2}, \frac{r}{2}$. We can choose to express this constant sequence in the form $(r-1)g+((r+1)g+1), 2(r-1)g+(2g+1), 2(r-1)g+(2g+1), \ldots, 2(r-1)g+(2g+1), (r-1)g+((r+1)g+1)$.

The first terms in each expression above can be regarded as forces acting on the $(r-1)$-slab contained in the r-slab, which then, by the induction hypothesis, yield downward forces on the bottom row of $rg+r-1, 2rg+2(r-1), \ldots, 2rg+2(r-1), rg+r-1$ at positions $-\frac{r-1}{2}, -\frac{r-3}{2}, \ldots, \frac{r-3}{2}, \frac{r-1}{2}$.

The second terms from the sequence, together with the weights of the outermost blocks of the $(r+1)$-rows, are passed straight down through the rigid structure of the $(r-1)$-slab to the bottom row. Now the combined forces acting down on the bottom row are $(r+1)g+r-1, rg+r-1, 2g+1, 2rg+2(r-1), 2g+1, \ldots, 2g+1, rg+r-1, (r+1)g+r-1$, at positions $-\frac{r}{2}, -\frac{r-1}{2}, \ldots, \frac{r-1}{2}, \frac{r}{2}$. The bottom row is in equilibrium with the given upward forces as required. $\square$

LEMMA 4.2. *For any $d \geqslant 1$, a d-stack is stable, contains $(d+1)d(2d+1)/3+1$ blocks and has an overhang of $(d+1)/2$.*

Proof. The number of blocks in a d-stack is $1 + \sum_{r=1}^d 2r^2 = 1 + \frac{(d+1)d(2d+1)}{3}$. Equilibrium follows by repeated application of Lemma 4.1 with corresponding values of $g = g(r)$ given by $g(r) = \frac{1}{r(r+1)}\sum_{i=r+1}^d i^2$.

Note that $g(d) = 0$. For compatibility, the forces at the bottom of the r-slab must equal the forces at the

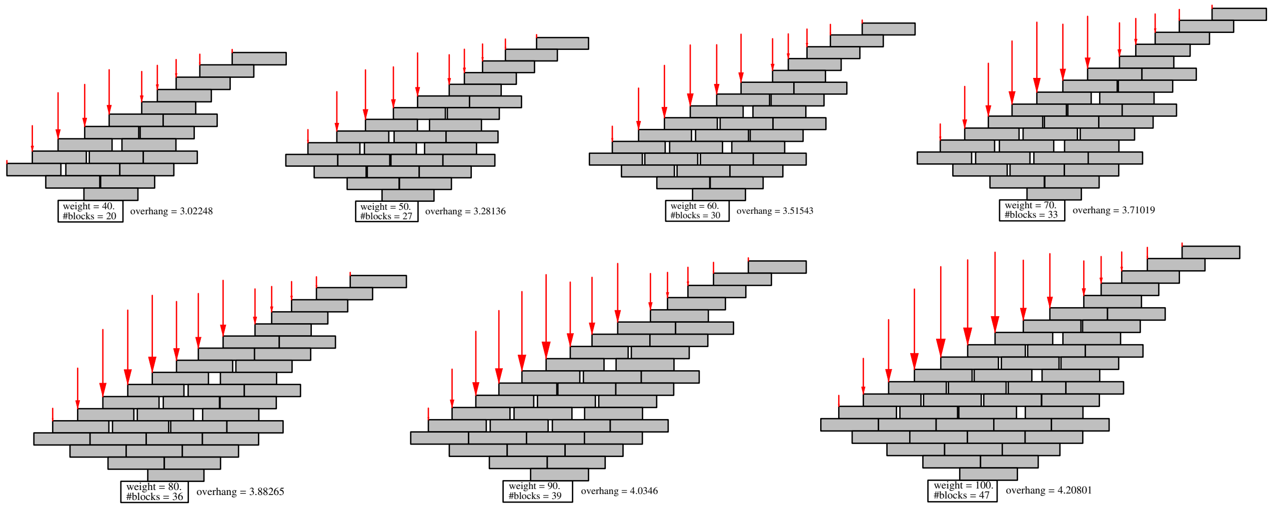

Figure 9: Loaded constructions that we believe to be optimal with total weights of $40, 50, \ldots, 100$. The constructions become more and more non-spinal. The lengths of the arrows depicting the external forces are proportional to their magnitude. The height of a block corresponds to the weight of a block.

top of the $(r-1)$-slab, for $r = d, d-1, \ldots, 2$. This holds, since

$$(r+1)g(r)+r = \frac{1}{r}\sum_{i=r+1}^{d} i^2+r = \frac{1}{r}\sum_{i=r}^{d} i^2 = (r-1)g(r-1).$$

□

Proof. (of Theorem) If the total number of blocks n satisfies $d(d-1)(2d-1)/3+1 \leqslant n \leqslant (d+1)d(2d+1)/3$ for some positive integer d, then Lemma 4.2 shows that a $(d-1)$-stack yields an overhang of $d/2$ and can be constructed using n blocks. Any extra blocks can be just placed in a vertical pile in the centre on top of the stack without disturbing stability. Hence

$$n < 2(d+\tfrac{1}{2})^3/3 \ \text{ and so } \ D(n) = d/2 > \left(\tfrac{3n}{16}\right)^{\frac{1}{3}} - \tfrac{1}{4}.$$

□

By comparing this lower bound for $D(n)$ with the upper bound of $S(n) < 1 + \ln n$ from Corollary 3.1, we can verify that *no* spinal stack can be optimal when $n \geqslant 5000$.

5 General constructions

We have complemented our upper and lower bound theorems with extensive empirical investigations using Matlab and Mathematica. The results inform our conjectures and suggest possibilities for improving our theoretical bounds.

In most of our experiments we have concentrated on optimising $D^*(n)$, i.e., overhangs of *loaded* stacks, rather than $D(n)$, since we believe these to be very closed related and the increased continuity offered by the loadings makes optimisation more tractable and appears to give a smoother outcome.

For the range $30 < n \leqslant 100$ we think that we have found optimal stacks, though we do not have formal justification. Pictures of some of these stacks are shown in Figure 9. We have some larger constructions, e.g., in Figures 10,11,13, which we believe are close to optimal.

In Section 4, we gave a lower bound for "brick-wall" constructions. Under this restriction, the number of possible stacks for any particular n is finite and we can greatly extend the range for an exhaustive search. On the grounds of simplicity and aesthetics we have investigated symmetric (about $x = 0$) stacks for values of n up to over 100,000. An interesting outcome of these experiments is that the shape of optimal symmetric stacks, after suitable scaling, seems to tend to a limit curve. See Figures 10 and 13 for examples we have computed. The shape of the curve, which we have termed the *vase*, is similar but different to that of the normal distribution. We have as yet no conjecture for its equation.

Asymmetric brickwall stacks can achieve slightly better overhang but the limiting behaviour is more difficult to interpret. The stack shown in Figure 11 reaches the same overhang as the symmetric stack in Figure 10 but requires about 3.5% less weight.

6 Upper bounds

In this section we prove upper bounds for the overhang reachable by a stack with n blocks. The most general upper bound we establish is $O(n^{1/2}\log n)$ (Theorem 6.1), but we can improve this to $O(n^{1/3}\log n)$ (Theorem 6.2) under a (plausible) assumption about optimal

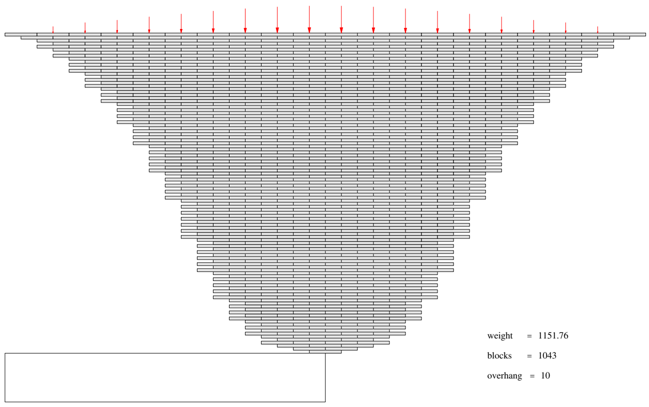

Figure 10: A symmetric loaded brick wall construction with an overhang of 10 and a total weight of about 1152. This is the best symmetric brick wall construction that we have found.

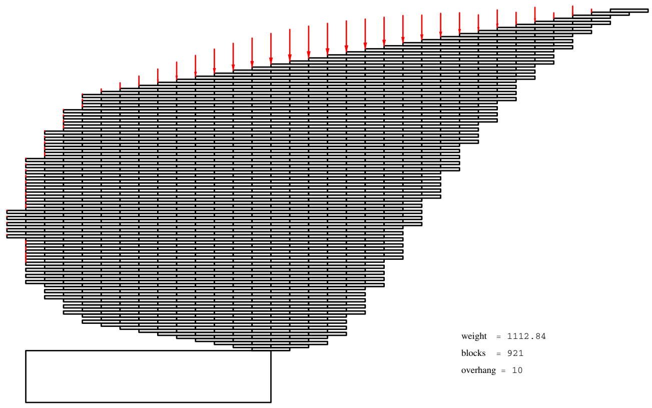

Figure 11: An asymmetric loaded brick wall construction with an overhang of 10 and a total weight of about 1113. This is the best asymmetric brick wall construction that we have found. It needs about 3.5% less weight for the same overhang as the symmetric stack in Figure 10.

stacks which is described below.

In our experiments to find optimal stacks for small values of n and in our best systematic constructions to date, we have found that the upper contour of the stack (inclusive of the balancing set) can be made to be non-decreasing from the right to at least as far left as the tabletop. The lower contour must decrease from right to left to the tabletop, otherwise some blocks would be unsupported. Furthermore, although there are sometimes internal gaps between blocks in the same row, we have never needed gaps immediately one above the other. These features are captured in the definition of "γ-dense" below. Theorem 6.2 assumes this property.

Consider a stack with n blocks, overhang D and height h. Rows are labelled from bottom to top as $Row_1, \ldots, Row_h$. Rows are regarded as sets of blocks. For convenience, we may consider the table as Row_0, tiled with adjacent blocks up to the edge. Columns are labelled from right (furthest overhang) to left. Col_j is the set of blocks with righthand edge in the range $(D-j, D-j+1]$, where the table edge is at 0. Note that each row from 1 to h is nonempty, as is each column from the rightmost column Col_1 to at least Col_w, where $w = \lfloor D+1 \rfloor$.

There is at most one block in the intersection of a row and a column. If the intersection of Row_i and Col_j is nonempty, denote this block as $B_{i,j}$.

Definitions 7. *Let $C_j = \{i \mid Row_i \cap Col_j \neq \emptyset\}$, so C_j lists all rows with a block in Col_j. Let $c_j = |C_j| = |Col_j|$ for all j. For any k, $1 \leqslant k \leqslant w$, we define $C_{\leqslant k} = \bigcup_{1 \leqslant j \leqslant k} C_j$ and $c_{\leqslant k} = |C_{\leqslant k}|$. Then a stack is* γ-dense *if $c_k \geqslant \gamma c_{\leqslant k}$ for $1 \leqslant k \leqslant w$.*

Theorem 6.1. *$D(n) = O(n^{1/2} \log n)$.*

It is convenient to defer the proof of this theorem until after the proof of Theorem 6.2, which is similar but more involved.

Theorem 6.2. *For any fixed $\gamma > 0$, the maximum overhang for any γ-dense stack of n blocks is $O(n^{1/3} \log n)$.*

Note that the brickwall stacks that we constructed in Section 4 are 1-dense.

Proof. Consider any γ-dense stack of n blocks with overhang D. For all i, j, let $d_{i,j}$ be the force exerted down on $B_{i,j}$ by $B_{i+1,j+1}$, if they both exist, and 0 otherwise. Let $u_{i,j}$ be the total force exerted by any blocks upwards on $B_{i,j}$, if it exists, and 0 otherwise.

We define $Col_{\leqslant j} = \bigcup_{i=1}^{j} Col_i$ and consider the balance of forces on $Col_{\leqslant j}$. In particular we examine the turning moments about the coordinate $D-j$. The external moment from the rest of the stack acting on $Col_{\leqslant j}$ is the sum of such moments acting on $B_{i,j}$ for all i. If $B_{i,j}$ exists, the anticlockwise moment about $D-j$ acting on $B_{i,j}$ is at most $d_{i,j}$. The anticlockwise moment due to gravity on the weight of $B_{i,j}$ is at most $1/2$. (See Figure 12.) These limits could only be approached if the lefthand edge of $B_{i,j}$ were close to coordinate $D-j-1$ and the external downward force $d_{i,j}$ from $B_{i+1,j+1}$ acted close to this edge. We will combine these two terms using the inequality: $d_{i,j} + 1/2 < d_{i,j} + 1 \leqslant u_{i,j}$ if block $B_{i,j}$ exists. The clockwise moment due to gravity acting on some block $B_{u,v}$ in $Col_{\leqslant j}$ is at least $j - v - 1/2 > 0$, for $1 \leqslant v < j$.

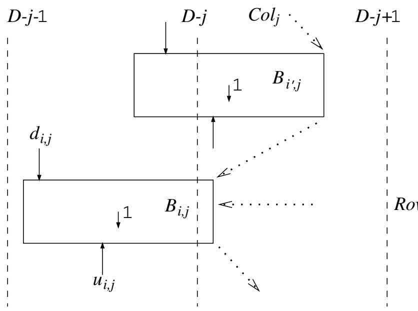

Figure 12: The forces on block $B_{i,j}$

For the stability of $Col_{\leqslant j}$, where $1 \leqslant j \leqslant w$, the following inequality must hold:

$$(*) \quad \sum_{i \in C_j} u_{i,j} > \sum_{i \in C_j} (d_{i,j} + \tfrac{1}{2}) \geqslant \tfrac{1}{2}c_{j-1} + \tfrac{3}{2}c_{j-2} + \cdots + \tfrac{2j-3}{2}c_1.$$

We also have for each i the "total load" inequality $\sum_j u_{i,j} \leqslant n$.

For some fixed k, where $1 \leqslant k \leqslant w$, we now prepare to sum each side of inequality $(*)$ for $1 \leqslant j \leqslant k$. Recall that $C_{\leqslant k} = \bigcup_{1 \leqslant j \leqslant k} C_j$ and $c_{\leqslant k} = |C_{\leqslant k}|$. From the lefthand side of $(*)$,

$$\sum_{1 \leqslant j \leqslant k} \sum_{i \in C_j} u_{i,j} \leqslant \sum_{1 \leqslant j \leqslant k} \sum_{i \in C_{\leqslant k}} u_{i,j} = \sum_{i \in C_{\leqslant k}} \sum_{1 \leqslant j \leqslant k} u_{i,j} \leqslant \sum_{i \in C_{\leqslant k}} n = nc_{\leqslant k} \ .$$

From the righthand side of $(*)$,

$$\begin{aligned} &\sum_{1 \leqslant j \leqslant k} \tfrac{1}{2}(c_{j-1} + 3c_{j-2} + \cdots + (2j-3)c_1) \\ &= \tfrac{1}{2} \sum_{1 \leqslant r < k} ((2r-1) + (2r-3) + \cdots + 3 + 1)c_{k-r} \\ &= \sum_{1 \leqslant r < k} \tfrac{r^2}{2} c_{k-r}. \end{aligned}$$

Since the stack is γ-dense, we have $c_k \geqslant \gamma c_{\leqslant k}$ and obtain the recurrence inequality:

$$nc_k \geqslant \sum_{1 \leqslant r < k} \tfrac{\gamma r^2}{2} c_{k-r}$$

for $1 \leqslant k \leqslant w$. Recall that $w = \lfloor D + 1 \rfloor$.

LEMMA 6.1. *Suppose that for $1 \leqslant k \leqslant w$, we have $c_k \geqslant 1$ and*

$$c_k \geqslant \sum_{1 \leqslant r < k} \tfrac{\gamma r^2}{2n} c_{k-r}.$$

Let $m = \lceil (n/\gamma)^{1/3} \rceil$, $\alpha = 1 + 1/m$, $M = \lceil 2m \ln m \rceil$, and $\beta = \alpha^{-M}$. Then $c_k \geqslant \beta\alpha^k$ for $1 \leqslant k \leqslant w$.

Proof. We establish that $c_k \geqslant \beta\alpha^k$ by induction on k. For $1 \leqslant k \leqslant M$, $c_k \geqslant 1 = \beta\alpha^M \geqslant \beta\alpha^k$.

For $M + 1 \leqslant k \leqslant w$,

$$c_k \geqslant \sum_{1 \leqslant r \leqslant k-1} \frac{\gamma r^2}{2n} c_{k-r} \geqslant \frac{\beta\alpha^k}{2n/\gamma} \sum_{1 \leqslant r \leqslant k-1} r^2 \alpha^{-r}.$$

It may be verified that

$$\begin{aligned} \sum_{r=1}^{k-1} r^2 \alpha^{-r} &= \frac{\alpha}{(\alpha-1)^3} \left(\alpha + 1 - \frac{(k(\alpha-1)+1)^2 + 2}{\alpha^k} \right) \\ &> m^3 \left(2 + 1/m - \frac{(k/m+1)^2 + 2}{\alpha^k} \right). \end{aligned}$$

Now, since k^2/α^k is a decreasing function of k for $k > 2/\ln\alpha \sim 2m$ and $k > M = \lceil 2m \ln m \rceil$, we deduce that for sufficiently large values of n

$$\begin{aligned} \frac{(k/m+1)^2 + 2}{\alpha^k} &< \frac{(M/m+1)^2 + 2}{(1+1/m)^M} \\ &\sim 4(\ln m)^2 e^{-2\ln m} = 4(\ln m)^2/m^2. \end{aligned}$$

Hence

$$\sum_{r=1}^{k-1} r^2 \alpha^{-r} = m^3(2 + 1/m - o(1/m)) > 2n/\gamma,$$

for sufficiently large n. So we have shown that $c_k \geqslant \alpha\beta^k$, proving the induction step. □

We now complete the proof of Theorem 6.2. From the inequalities

$$n \geqslant \sum_{k=1}^{w} c_k \geqslant \beta \sum_{k=1}^{w} \alpha^k \sim \beta\alpha^D/(\alpha - 1) = \alpha^{D-M} m,$$

we derive $(D - M)\ln\alpha \leqslant \ln(n/m)$, and therefore

$$\begin{aligned} D &\leqslant \tfrac{\ln(n/m)}{\ln\alpha} + M \leqslant m\ln(n/m) + M \\ &\leqslant (2n/\gamma)^{1/3} \ln(n^{2/3}) + O(n^{1/3}\ln n) \\ &= O(n^{1/3}\log n). \end{aligned}$$

□

To finish this section, we give the proof of the more general upper bound.

Proof. (Theorem 6.1) The proof begins as in the proof of Theorem 6.2 up to inequality $(*)$, but here we just use the naive inequality $d_{i,j} + \frac{1}{2} < n$, to derive the weaker recurrence:

$$\begin{aligned} nc_j &> \sum_{i \in C_j} (d_{i,j} + \tfrac{1}{2}) \\ &\geqslant \tfrac{1}{2}c_{j-1} + \tfrac{3}{2}c_{j-2} + \cdots + \tfrac{2j-3}{2}c_1 \\ &= \tfrac{1}{2}\sum_{r=1}^{j-1}(2r-1)c_{j-r}. \end{aligned}$$

Corresponding to Lemma 6.1, we now have:

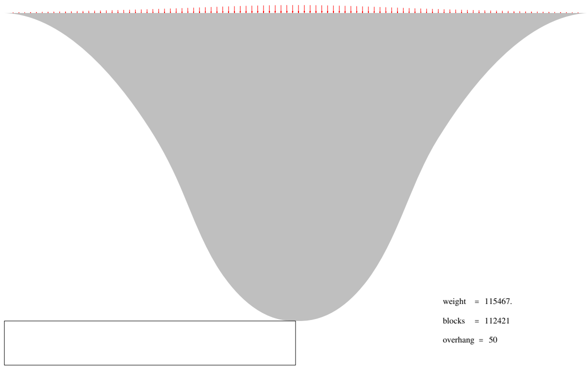

Figure 13: A scaled outline of a loaded brick wall construction with an overhang of 50 and a total weight of about 115, 467. This is the best symmetric brick wall construction that we have found. Individual blocks are not shown as they are too small to be visible.

LEMMA 6.2. *Suppose that for* $1 \leqslant k \leqslant w$, *we have* $c_k \geqslant 1$ *and*

$$c_k \geqslant \sum_{1\leqslant r<k} \tfrac{2r-1}{2n} c_{k-r}.$$

Let $m = \lceil (2n)^{1/2} \rceil$, $\alpha = 1 + 1/m$, $M = \lceil 2m \ln m \rceil$, *and* $\beta = \alpha^{-M}$. *Then* $c_k \geqslant \beta\alpha^k$ *for* $1 \leqslant k \leqslant w$.

Proof. The inductive proof is similar to before, but now, for $M + 1 \leqslant k \leqslant w$, we have

$$c_k \geqslant \sum_{1\leqslant r\leqslant k-1} \frac{2r-1}{2n} c_{k-r} \geqslant \frac{\beta\alpha^k}{2n} \sum_{1\leqslant r\leqslant k-1} (2r-1)\alpha^{-r}.$$

The summation may be explicitly evaluated as

$$\sum_{r=1}^{k-1} (2r-1)\alpha^{-r} = \frac{1 + \frac{1}{\alpha} - \alpha^{-k+1}((2k-1)\alpha + (2k-3))}{(\alpha-1)^2}.$$

Hence,

$$\begin{aligned} \sum_{r=1}^{k-1} (2r-1)\alpha^{-r} &> \frac{1 + \frac{1}{\alpha} - \alpha^{-k+1} 2k(\alpha+1)}{(\alpha-1)^2} \\ &> \frac{1 + \frac{1}{\alpha} - \alpha^{-M+1} 2M(\alpha+1)}{(\alpha-1)^2}, \end{aligned}$$

since $k\alpha^{-k}$ is a decreasing function of k for $k > M$. Now,

$$\begin{aligned} \alpha^{-M+1} M(\alpha+1) &\sim (1 + 1/m)^{-2m \ln m} m \ln m \\ &\sim e^{-2lnm} m \ln m = (\ln m)/m = o(1). \end{aligned}$$

Therefore

$$\sum_{r=1}^{k-1} (2r-1)\alpha^{-r} > (1 + 1/\alpha - o(1))m^2 > 2n$$

for sufficiently large n. So we have shown that $c_k \geqslant \alpha\beta^k$, proving the induction step. $\square$

To complete the proof of Theorem 6.1, we proceed in a similar way to before, and derive

$$\begin{aligned} D &\leqslant \tfrac{\ln(n/m)}{\ln \alpha} + M \leqslant m \ln(n/m) + M \\ &\leqslant (2n)^{1/2} \ln(n^{1/2}) + O(n^{1/2} \ln n) = O(n^{1/2} \log n). \end{aligned}$$

$\square$

7 Concluding remarks and open problems

We have revisited a well-known classic problem and begun to answer some of the questions that were latent there. We have shown that the overhang achievable with n blocks is exponentially larger than was previously supposed. We believe that our constructions here are asymptotically close to optimal.

Our upper bound of $O(n^{1/3} \log n)$ (Theorem 6.2) is only proved under a restriction on the structure of stacks. To improve our general upper bound of $O(n^{1/2} \log n)$ we would need to show that optimal stacks did not need large internal voids in their structure.

Our empirical experiments for large n suggest that a "vase" shape may be optimal. The contours of our best symmetric brick wall constructions with overhangs of 10 (Figure 10), 50 (Figure 13), 100 and 200, after appropriate scaling, are similar to a Gaussian curve, but we do not have a conjecture for the real function approximating this contour.

The major open problem, however, is to resolve the following conjecture:

CONJECTURE 3. $D(n) = \tilde{\Theta}(n^{1/3})$.

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