Minimum Bisection and Spectral Bisection.

<u>Minimum Bisection Problem</u>: Given an n-vertex graph G, partition $V(G)$ into two sets A_1 and A_{-1} such that

$$\underbrace{\big||A_1| - |A_{-1}|\big|}_{|A_1| = \lceil \frac{n}{2} \rceil \quad |A_{-1}| = \lfloor \frac{n}{2} \rfloor} \leq 1$$

and

$$e_G(A_1, A_{-1})$$

$\leftarrow$ #edges between A_1 and A_{-1}.

$|\{vw \in E(G) : v \in A_1,\ w \in A_{-1}\}|$

is minimized.

MBP is NP-hard, there is probably no algorithm for MBP whose running time is $O(n^c)$ for any constant c.

If we could solve it, MBP might be useful as part of a top-down approach to community detection.

- Find best partition A_1, A_{-1}
  - recurse on $G[A_1]$.
  - recurse on $G[A_{-1}]$.

First Try

- Find column n-vector $x \in \{1,-1\}^n$ that minimizes

$$xLx^T = 4 \cdot \left(\#\text{edges between } A_1 \text{ and } A_{-1}\right) \qquad (\star)$$

- Problem: Minimized when $x = \mathbf{1}$ or $x = -\mathbf{1}$

$\mathbf{1}$: n-vector of 1's. $-\mathbf{1}$: n-vector of -1's.

because we can put everything in A_1 or everything in A_{-1} and then no edges cross the partition.

- Solution: Add the constraint $\left|\sum_{i=1}^{n} x_i\right| \leq 1$. $\qquad (*)$

Then $\big||A_1| - |A_{-1}|\big| \leq 1$, so the two parts have almost exactly the same size.

Bigger Problem: Finding $x \in \{1,-1\}^n$ that satisfies $(*)$ and minimizes $(\star)$ is NP-hard. This is called the minimum bisection problem.

Notes on Linear Algebra Problem.

L is symmetric and semi-positive definite.

$\Rightarrow$. All eigenvalues of L are non-negative and all eigenvectors are orthogonal.

Smallest eigenvalue: 0, corresponding to the vector $x_0 = \mathbb{1}$

$$L x_0 = 0 \cdot x_0$$

Second smallest eigenvalue λ_2, solves.

$$L x_2 = \lambda_2 x_2$$

Why is this a good solution?

The eigenvector x_2 is orthogonal to $x_0 = \mathbb{1}$,

so $x_2 \cdot x_0 = x_2 \cdot \mathbb{1} = 0$, so $\sum_{i=1}^{n} (x_2)_i = 0$.

~~Then the~~ So x_2 has positive and negative entries and the partition is non-trivial.

λ

Minimum Bisection Problem.

- Best Known Approximation ratio is $O((\log n)^{1.5})$.
  - There exists a polynomial-time algorithm that finds a partition A, B with $||A|-|B|| \leq 1$ s.t.

$$e(A,B) \leq e(A^*, B^*) \cdot c(\log n)^{1.5}$$

- where A^*, B^* is a partition that minimizes $e(A^*, B^*)$ among all partitions A^*, B^* with $||A^*|-|B^*|| \leq 1$.