Carleton University

Final
Examination
Fall 2014

DURATION: 2 HOURS — No. of students: 136

Department Name & Course Number: **Computer Science COMP 2804A**
Course Instructor: Michiel Smid

Authorized memoranda:
NONE

Students MUST count the number of pages in this examination question paper before beginning to write, and report any discrepancy to the proctor. This question paper has 11 pages (not including the cover page).

This examination question paper MAY be taken from the examination room.

In addition to this question paper, students require:

an examination booklet:	**no**
a Scantron sheet:	**yes**

Instructions:

1. This is a closed book exam. No aids, notes, or calculating devices are allowed.
2. All questions must be answered on the scantron sheet.

Marking scheme: Each of the 25 questions is worth 1 mark.

- Newton: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$.
- Geometric distribution: Assume an experiment has a success probability of p. We perform the experiment until it is successful for the first time. The expected number of times we perform the experiment is $1/p$.

1. A password consists of 13 characters, each character being one of the ten digits $0, 1, 2, \ldots, 9$. A password must contain exactly one odd digit. How many passwords are there?

   (a) $13 \cdot 5^{12}$
   (b) $13 \cdot 5^{13}$
   (c) $13 \cdot 9^{12}$
   (d) $13 \cdot 5 \cdot 9^{12}$

2. A password consists of 13 characters, each character being one of the ten digits $0, 1, 2, \ldots, 9$. A password must contain at least one odd digit and at most two even digits. How many passwords are there?

   (a) $5^{12} + 13 \cdot 5^{12} + \binom{13}{2} 5^{12}$
   (b) $5^{13} + 13 \cdot 9^{12} + \binom{13}{2} 9^{12}$
   (c) $5^{13} + 13 \cdot 5^{13} + \binom{13}{2} 5^{13}$
   (d) $5 \cdot 9^{12} + 13 \cdot 5 \cdot 9^{12} + \binom{13}{2} 5 \cdot 9^{12}$

3. What is
$$\sum_{k=0}^{45} \binom{45}{k} (-3)^{2k}.$$

   (a) $(-2)^{45}$
   (b) 4^{45}
   (c) $(-8)^{45}$
   (d) 10^{45}

4. How many bitstrings of length 13 start with 010 or end with 11?

   (a) $2^{10} + 2^{11}$
   (b) $3 \cdot 2^{10} - 2^8$
   (c) $2^{13} - (2^{10} + 2^{11})$
   (d) None of the above.

5. Let S_n be the number of bitstrings of length n that do not contain the substring 11. Which of the following is true?

   (a) $S_n = S_{n-1} + S_{n-2}$ for $n \geq 3$.
   (b) $S_n = S_{n-1} + S_{n-3}$ for $n \geq 3$.
   (c) $S_n = S_{n-1} + S_{n-2} + S_{n-3}$ for $n \geq 3$.
   (d) $S_n = 2 \cdot S_{n-1}$ for $n \geq 3$.

6. In a group of 100 students,

   - 40 like 8:30am classes,
   - 30 like the course COMP 2804,
   - 50 do not like 8:30am classes and do not like the course COMP 2804.

   How many students in this group like 8:30am classes and like the course COMP 2804?

   (a) 10
   (b) 20
   (c) 30
   (d) 40

7. Consider $m \geq 7$ blue balls $B_1, B_2, \ldots, B_m$ and $n \geq 7$ red balls $R_1, R_2, \ldots, R_n$. We pick 7 balls of the same color and arrange them on a horizontal line. (The order on the line matters.) How many arrangements are there?

   (a) $7!\binom{m}{7} + 7!\binom{n}{7}$
   (b) $m!\binom{m}{7} + n!\binom{n}{7}$
   (c) $7!\binom{m+n}{7}$
   (d) None of the above.

8. The number of different strings that can be made by reordering the 10 letters of the word AAABBCCCCC is

   (a) 10!
   (b) $\frac{10!}{2!3!5!}$
   (c) $\binom{10}{3}\binom{10}{2}\binom{10}{5}$
   (d) None of the above.

9. How many solutions are there to the equation $x_1 + x_2 + x_3 = 13$, where $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$ are integers?

   (a) $\binom{13}{2}$
   (b) $\binom{14}{2}$
   (c) $\binom{15}{2}$
   (d) $\binom{16}{2}$

10. Consider the following recursive function:

$$\begin{aligned} f(0) &= -17, \\ f(n) &= f(n-1) + 8n - 2 \text{ for all integers } n \geq 1. \end{aligned}$$

Which of the following is true?

(a) for all $n \geq 0$: $f(n) = 3n^2 - n - 17$

(b) for all $n \geq 0$: $f(n) = 3n^2 + n - 17$

(c) for all $n \geq 0$: $f(n) = 4n^2 - 2n - 17$

(d) for all $n \geq 0$: $f(n) = 4n^2 + 2n - 17$

11. Consider the recursive algorithm Fib, which takes as input an integer $n \geq 0$:

Algorithm Fib(n):
if $n = 0$ or $n = 1$
then $f = n$
else $f =$ Fib$(n-1)$ + Fib$(n-2)$
endif;
return f

If we run algorithm Fib(20), how many calls are there to Fib(16)?

(a) 4

(b) 5

(c) 6

(d) 7

12. A bowl contains 5 blue balls, 4 red balls, and 6 green balls. We choose 2 balls uniformly at random. What is the probability that these 2 balls have the same color?

(a) $\frac{\binom{5}{2}+\binom{4}{2}+\binom{6}{2}}{\binom{15}{2}}$

(b) $\frac{\binom{15}{2}}{\binom{5}{2}+\binom{4}{2}+\binom{6}{2}}$

(c) $\frac{\binom{5}{2}\binom{4}{2}\binom{6}{2}}{\binom{15}{2}}$

(d) $\frac{\binom{15}{2}}{\binom{5}{2}\binom{4}{2}\binom{6}{2}}$

13. Annie, Boris, and Charlie have random and independent birthdays. (We ignore leap years, so that a year has 365 days.) What is the probability that Annie, Boris, and Charlie have the same birthday?

    (a) $\frac{1}{364 \cdot 365}$
    (b) $\frac{1}{365^2}$
    (c) $\frac{365}{364^2}$
    (d) $\frac{1}{365^3}$

14. We flip a fair coin repeatedly and independently, resulting in a sequence of heads (H) and tails (T). We stop flipping the coin as soon as this sequence contains one H or eight Ts. What is the probability that this sequence contains at most seven Ts?

    (a) $1 - (1/2)^7$
    (b) $\sum_{k=0}^{7} (1/2)^k$
    (c) $\sum_{k=0}^{7} (1/2)^{k+1}$
    (d) None of the above.

15. A bowl contains 5 blue balls and 4 red balls. We choose 2 balls uniformly at random. Define the events

$$A = \text{"both balls are red"},$$

$$B = \text{"both balls have the same color"}.$$

    What is the conditional probability $\Pr(A \mid B)$?

    (a) $\frac{\binom{4}{2}}{\binom{5}{2}+\binom{4}{2}}$
    (b) $\frac{\binom{5}{2}+\binom{4}{2}}{\binom{4}{2}}$
    (c) $\frac{\binom{4}{2}}{\binom{9}{2}}$
    (d) $\frac{\binom{4}{2}}{\binom{5}{2}\binom{4}{2}}$

16. We choose an element x uniformly at random from the set $\{1, 2, 3, \dots, 10\}$. Define the events

$$A = \text{"}x\text{ is even"},$$

$$B = \text{"}x\text{ is divisible by 3"}.$$

What is the conditional probability $\Pr(A \mid B)$?

(a) 3/10
(b) 1/3
(c) 1/2
(d) 2/3

17. We choose an element x uniformly at random from the set $\{1, 2, 3, \dots, 10\}$. Define the events

$$A = \text{"}x\text{ is even"},$$

and

$$B = \text{"}1 \leq x \leq 5\text{"}.$$

Which of the following is true?

(a) The events A and B are independent.
(b) The events A and B are not independent.
(c) None of the above.

18. We choose an element x uniformly at random from the set $\{1, 2, 3, \dots, 10\}$. Define the events

$$A = \text{"}x\text{ is even"},$$

and

$$B = \text{"}1 \leq x \leq 6\text{"}.$$

Which of the following is true?

(a) The events A and B are independent.
(b) The events A and B are not independent.
(c) None of the above.

19. When a couple has a child, this child is a boy with probability 1/2 and a girl with probability 1/2, independent of the gender of previous children. A couple stops having children as soon as they have a child that has the same gender as their first child. Define the events

$$A = \text{“the second child is a boy”},$$

and

$$B = \text{“the couple has at least three children and the third child is a boy”}.$$

Which of the following is true?

(a) The events A and B are independent.
(b) The events A and B are not independent.
(c) None of the above.

20. Assume you answer the first question in this exam by choosing one of the four answers uniformly at random. You answer the second question by choosing, again uniformly at random, one of the three answers you did not choose in the first question. What is the probability that you answer the second question correctly?

(a) $\frac{1}{3}$
(b) $\frac{1}{4}$
(c) $\frac{1}{4} \cdot \frac{1}{3} + \frac{3}{4} \cdot \frac{1}{4}$
(d) none of the above

21. I flip two fair and independent coins. If the first coin comes up tails, you lose \$1 (i.e., you win $-$\$1). If the second coin comes up heads, you win \$2. (Thus, if the first coin comes up tails and the second coins comes up heads, you win \$1.) Define the random variable X to be the amount of dollars that you win. What is the expected value of X?

(a) 2
(b) 1
(c) 1/4
(d) 1/2

22. I flip a fair coin, independently, 6 times, resulting in a sequence of heads (H) and tails (T). For each (consecutive) HTH in this sequence, you win \$5. Define the random variable X to be the amount of dollars that you win. For example, if the sequence is

$$THTHTH,$$

then $X = 10$. What is the expected value of X?

   (a) 2/5
   (b) 2
   (c) 5/2
   (d) 5

23. Consider the following recursive algorithm ThreeHeadsOrThreeTails, which takes as input a positive integer k:

   Algorithm ThreeHeadsOrThreeTails(k):
   // all coin flips made are mutually independent
   flip a fair coin three times;
   if the result is HHH or TTT
   then return k
   else ThreeHeadsOrThreeTails($k + 1$)
   endif

   You run algorithm ThreeHeadsOrThreeTails(1), i.e., with $k = 1$. Define the random variable X to be the value of the output of this algorithm. What is the expected value of X?

   (a) 2
   (b) 3
   (c) 4
   (d) 5

24. We flip a fair coin independently n times. Define the random variable

$$X = \text{ twice the number of heads minus three times the number of tails.}$$

What is the expected value of X?

(a) $-n/2$
(b) $n/2$
(c) $-n$
(d) n

25. Who invented the Fibonacci numbers?

(a) Justin Bieber
(b) Britney Spears
(c) Fibonacci
(d) Carl Friedrich Gauss