Carleton University

DURATION: 2 HOURS

No. of students: 152

Department Name & Course Number: **Computer Science COMP 2804B**
Course Instructor: Michiel Smid

Authorized memoranda:
NONE

Students MUST count the number of pages in this examination question paper before beginning to write, and report any discrepancy to the proctor. This question paper has 11 pages (not including the cover page).

This examination question paper MAY be taken from the examination room.

In addition to this question paper, students require:

        an examination booklet: no
        a Scantron sheet: yes

Instructions:

1. This is a closed book exam. No aids, notes, or calculating devices are allowed.

2. All questions must be answered on the scantron sheet.

Marking scheme: Each of the 25 questions is worth 1 mark.

- Newton: $(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$.

- Geometric distribution: Assume an experiment has a success probability of p. We perform the experiment until it is successful for the first time. The expected number of times we perform the experiment is $1/p$.

1. A password consists of 28 characters, each character being a lowercase letter. A password must contain exactly one vowel (i.e., a, e, i, o, or u). How many passwords are there?

    (a) $5 \cdot 21^{27}$

    (b) $28 \cdot 5 \cdot 21^{27}$

    (c) $28 \cdot 5 \cdot 27^{21}$

    (d) $28 \cdot 5 \cdot 26^{27}$

2. A password consists of 10 characters, each character being a lowercase letter or a digit. A password must contain at least one digit and at most three digits. How many passwords are there?

    (a) $\binom{10}{1}26^9 + \binom{10}{2}26^8 + \binom{10}{3}26^7$

    (b) $10 \cdot 26^9 + 10^2 \cdot 26^8 + 10^3 \cdot 26^7$

    (c) $\binom{10}{1} \cdot 10 \cdot 26^9 + \binom{10}{2} \cdot 10^2 \cdot 26^8 + \binom{10}{3} \cdot 10^3 \cdot 26^7$

    (d) none of the above

3. What is
$$\sum_{k=0}^{45} \binom{45}{k}(-3)^{2k}.$$

    (a) $(-2)^{45}$

    (b) 4^{45}

    (c) $(-8)^{45}$

    (d) 10^{45}

4. Consider a group of 8 people, consisting of Simon, John, and 6 other people. How many ways are there to arrange these 8 people on a horizontal line such that Simon and John are standing next to each other? (The order on the line matters and Simon is either to the left or to the right of John.)

    (a) $7 \cdot 6!$

    (b) $2 \cdot 7 \cdot 6!$

    (c) $8 \cdot 6!$

    (d) $2 \cdot 8 \cdot 6!$

5. Consider strings consisting of n characters, each character being a, b, or c. Let S_n be the number of such strings of length n that do not contain the substring aa. Which of the following is true?

   (a) $S_{n+1} = S_n + S_{n-1}$ for $n \geq 2$.

   (b) $S_{n+1} = 2 \cdot S_n + S_{n-1}$ for $n \geq 2$.

   (c) $S_{n+1} = S_n + 2 \cdot S_{n-1}$ for $n \geq 2$.

   (d) $S_{n+1} = 2 \cdot S_n + 2 \cdot S_{n-1}$ for $n \geq 2$.

6. In a group of 150 people,

   - 30 weight at least 150 pounds,

   - 60 are at least 6 feet tall,

   - 70 weigh less than 150 pounds and are less than 6 feet tall.

   How many people weigh at least 150 pounds and are at least 6 feet tall?

   (a) 10

   (b) 20

   (c) 30

   (d) 40

7. Consider 4 blue balls B_1, B_2, B_3, B_4 and 5 red balls R_1, R_2, R_3, R_4, R_5. We pick 3 balls of the same color and arrange them on a horizontal line. (The order on the line matters.) How many arrangements are there?

   (a) 64

   (b) 74

   (c) 84

   (d) 94

8. Consider a group of n people, let k be an integer with $1 \leq k \leq n$, and consider a circular table with k chairs around it. We select k people and seat them around this table. How many different seating arrangements are there? (Two seating arrangements A and B are the same if for each person, the clockwise neighbor in A is the same as the clockwise neighbor in B, and the counterclockwise neighbor in A is the same as the counterclockwise neighbor in B.)

   (a) $\frac{n!}{k(n-k)!}$

   (b) $\frac{n!}{k!(n-k)!}$

   (c) $\frac{n!}{(n-k)!}$

   (d) $\frac{2 \cdot n!}{(n-k)!}$

9. How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 777$, where $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$, $x_4 \geq 0$ are integers?

   (a) $\binom{780}{3}$

   (b) $\binom{780}{4}$

   (c) $\binom{781}{3}$

   (d) $\binom{781}{4}$

10. Consider the following recursive function:

$$\begin{aligned} f(0) &= 3, \\ f(n+1) &= f(n) + 10n + 2 \text{ for all integers } n \geq 0. \end{aligned}$$

   Which of the following is true?

   (a) for all $n \geq 0$: $f(n) = 5n^2 - 3n + 2$

   (b) for all $n \geq 0$: $f(n) = 5n^2 - 3n + 3$

   (c) for all $n \geq 0$: $f(n) = 5n^2 + 3n + 3$

   (d) for all $n \geq 0$: $f(n) = 5n^2 - 2n + 3$

11. The Fibonacci numbers are defined as follows: $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$.

    Consider the recursive algorithm FIB, which takes as input an integer $n \geq 0$:

    > **Algorithm** FIB(n):
    > **if** $n = 0$ or $n = 1$
    > **then** $f = n$
    > **else** $f = $ FIB($n-1$) + FIB($n-2$)
    > **endif**;
    > return f

    For $n \geq 5$, run algorithm FIB(n) and let a_n be the number of times that FIB(4) is called. Which of the following is true?

    (a) for all $n \geq 5$, $a_n = f_n$

    (b) for all $n \geq 5$, $a_n = f_{n-1}$

    (c) for all $n \geq 5$, $a_n = f_{n-2}$

    (d) for all $n \geq 5$, $a_n = f_{n-3}$

12. We choose a subset of $\{1, 2, 3, 4, 5\}$ uniformly at random. What is the probability that this subset has size 3?

    (a) 4/16

    (b) 4/32

    (c) 5/16

    (d) 5/32

13. Let $A = \{1, 2, 3, \ldots, 100\}$. Let x, y, and z be elements in A that are chosen independently and uniformly at random. What is the probability that $x = y = z$?

    (a) $\frac{1}{100}$

    (b) $\frac{1}{100 \cdot 99}$

    (c) $\frac{1}{100 \cdot 100}$

    (d) $\frac{1}{\binom{100}{2}}$

14. We flip a fair coin repeatedly and independently, resulting in a sequence of heads (H) and tails (T). We stop flipping the coin as soon as this sequence contains H or $TTTT$. What is the probability that this sequence contains at most two Ts?

   (a) 4/8

   (b) 5/8

   (c) 6/8

   (d) 7/8

15. We roll two fair and independent dice. Define the events

$$A = \text{``at least one of the dice shows a 3''},$$

$$B = \text{``the sum of the dice is 5''}.$$

   What is the conditional probability $\Pr(A \mid B)$?

   (a) 1/2

   (b) 1/4

   (c) 1/8

   (d) 1/9

16. Let C_1 be a fair coin that has H on one side and T on the other side. Let C_2 be a coin that has H on both sides. We choose one of C_1 and C_2 uniformly at random and flip it. Define the events

$$A = \text{``we choose } C_2\text{''},$$

$$B = \text{``the flip resulted in } H\text{''}.$$

   What is the conditional probability $\Pr(A \mid B)$?

   (a) 1/3

   (b) 1/2

   (c) 2/3

   (d) none of the above

17. Let A and B be events in a sample space, such that $\Pr(A) = 1/3$, $\Pr(B) = 1/2$, and $\Pr(A \mid B) = 2/5$. What is $\Pr(B \mid A)$?

    (a) 1/5

    (b) 2/5

    (c) 3/5

    (d) 4/5

18. Consider a group of 5 people. Each person in this group was born in a uniformly random month (from the 12 months of the year), independent of the other people's month of birth. What is the probability that at least 2 of these 5 people were born in the same month?

    (a) $1 - \frac{12!}{12^4 \cdot 7!}$

    (b) $1 - \frac{11!}{12^4 \cdot 7!}$

    (c) $1 - \frac{11!}{12^4 \cdot 6!}$

    (d) $1 - \frac{11!}{12^4 \cdot 8!}$

19. Assume you answer the first question in this exam by choosing one of the four answers uniformly at random. You answer the second question by choosing, again uniformly at random, one of the three answers you did not choose in the first question. What is the probability that you answer the second question correctly?

    (a) $\frac{1}{3}$

    (b) $\frac{1}{4}$

    (c) $\frac{1}{4} \cdot \frac{1}{3} + \frac{3}{4} \cdot \frac{1}{4}$

    (d) none of the above

20. I flip two fair and independent coins. If the first coin comes up heads, you win \$1. If the second coin comes up heads, you win \$2. (Thus, if both coins come up heads, you win \$3.) Define the random variable X to be the amount of dollars that you win. What is the expected value of X?

    (a) 3

    (b) 2

    (c) 5/2

    (d) 3/2

21. When a couple has a child, this child is a boy with probability $1/2$ and a girl with probability $1/2$, independent of the gender of previous children. A couple stops having children as soon as they have 2 girls or 2 boys. Define the random variables

$$G = \text{ the number of girls the couple has,}$$

and

$$B = \text{ the number of boys the couple has.}$$

Are G and B independent random variables?

   (a) Yes

   (b) No

22. I flip a fair coin, independently, 10 times, resulting in a sequence of heads (H) and tails (T). For each HT in this sequence, you win \$3. Define the random variable X to be the amount of dollars that you win. For example, if the sequence is

$$HHTTHTTHTT,$$

then $X = 9$. What is the expected value of X?

   (a) $27/4$

   (b) $28/4$

   (c) $29/4$

   (d) $30/4$

23. We roll a pair of fair dice repeatedly and independently, and stop as soon as the sum of the numbers for the pair is 7. Define the random variable X to be the number of times we roll the dice. (In one roll, we roll a pair of dice.) What is the expected value of X?

   (a) 4

   (b) 5

   (c) 6

   (d) 7

24. We flip a fair coin independently n times. Define the random variable

$$X = \text{ the number of heads minus the number of tails in the sequence of } n \text{ flips.}$$

What is the expected value of X?

(a) $n/2$

(b) $n/4$

(c) $n/8$

(d) 0

25. Who discovered the formula

$$1 + 2 + 3 + \cdots + n = n(n+1)/2$$

when he was a young boy?

(a) Justin Bieber

(b) Fibonacci

(c) Carl Friedrich Gauss

(d) Simon Pratt