Theorem: Let S be a set of size n.
Then there are $n!$ permutations of S.

Proof: Product Rule OR observe that a permutation is a one-to-one function $f: \{1, \ldots, n\} \to S$.

$$\frac{n!}{\underbrace{(n-n)!}} = n!$$

$$0! = 1.$$

Binomial Coefficients: Let $n \geq 0$, $k \geq 0$ be integers. Then $\binom{n}{k}$ is the number of k-element subsets of a set of size n. – Definition!

Theorem: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Proof: (Combinatorial Proof, Counting two different ways).
- Let S be a set of size n.
- Let A be the set of ordered k-element subsets of S.

$$\binom{n}{k} \cdot k! \;\cancel{= |A|}\; = \frac{n!}{(n-k)!}$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

$S = \{a, b, c, d\}$ $n = 4$. $k = 2$.

2-element subsets: $\{a,b\}, \{a,c\}, \{a,d\}$. $\{b,c\}, \{b,d\}, \{c,d\}$.

$$\binom{4}{2} = 6.$$

ordered pairs: $\underbrace{(a,b), (b,a), (a,c)(c,a), (a,d), (d,a), (c,b), (b,c), (b,d)(d,b), (c,d), (d,c)}_{12.}$

1. Product Rule:
   (i) - choose a k-element subset of S. – $\binom{n}{k}$
   (ii) - choose an ordering of this subset. – $k!$
   $\binom{n}{k} \cdot k!$

2. For $i = 1$ to k.
   - choose the i^{th} element in the ordered subset.

$$n \cdot (n-1)(n-2) \cdots (n-(k-1))$$
$$= n(n-1)(n-2) \cdots (n-k+1) \cdot \frac{(n-k)(n-k-1) \cdots 1}{(n-k)(n-k-1) \cdots 1} = \frac{n!}{(n-k)!}$$

Example 1: A 5-card hand from a 52-card deck. $\binom{52}{5} = 2{,}598{,}960$.

Example 2: Bitstrings of length n with exactly k 1's. $\binom{n}{k}$.

- Product Rule: (i) Choose the locations of the k 1's and write 1's in those positions. $\rightarrow \binom{n}{k}$
  (ii) write 0's everywhere else. $\rightarrow 1$.

$\underline{0}\;\underline{1}\;\underline{0}\;\underline{1}\;\underline{1}\;\underline{0}$

Newton's Binomial Theorem: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$.

(FOIL)

$$(x+y)^2 = (x+y)(x+y) = x^2 + xy + xy + y^2 = x^2 + 2xy + y^2.$$

$$= \binom{2}{0} x^2 + \binom{2}{1} xy + \binom{2}{2} \cdot y^2$$

$$x^2 + 2xy + y^2.$$

Proof: $(x+y)^n = (x+y)(x+y)(x+y) \cdots\cdots (x+y)$.

$$x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots\cdots + \binom{n}{n} y^n$$

2.5

Example: What is the coefficient of $x^{12}y^{13}$ in $(2x-5y)^{25}$?

$n=25$.

$k=13$.

$$\big((2x)+(-5y)\big)^{25}$$

$$\binom{25}{13}(2x)^{12}\cdot(-5y)^{13} = \binom{25}{13}\cdot 2^{12}(-5)^{13}x^{12}y^{13}.$$

$$= -\binom{25}{13}2^{12}5^{13}x^{12}y^{13}$$

Theorem: $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} \cdot 1^k = \sum_{k=0}^{n} \binom{n}{k}$$

X = a n element set.

$P(X)$ is the power set of X

$$\sum_{k=0}^{n} \binom{n}{k} = |P(X)| = 2^n.$$

- $P_k(X)$ is the set of k-element subsets of X.

$$|P(X)| = |P_0(X) \cup P_1(X) \cup P_2(X) \cup \cdots \cup P_n(X)|$$

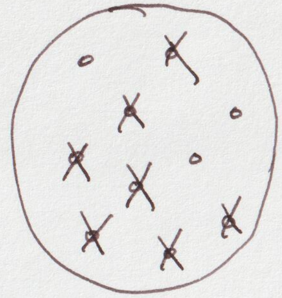
$$= |P_0(X)| + |P_1(X)| + \cdots + |P_n(X)|.$$

$$= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

$$= \sum_{k=0}^{n} \binom{n}{k}.$$

Theorem: $\binom{n}{k} = \binom{n}{n-k}$

S



$\binom{n}{n-k}$

(i) choose $n-k$ elements from S.

(ii) return all the unchosen elements.

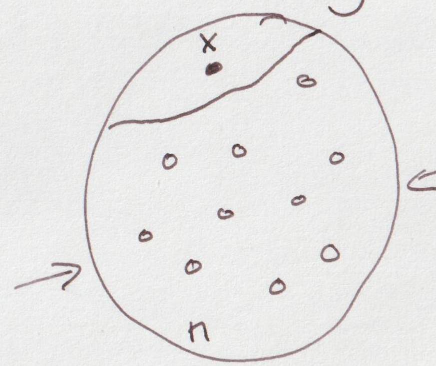
Proof: Let S be a n-element set.

Let A be the set of k-element subsets of S.

$$\binom{n}{k} = |A| = \binom{n}{n-k}$$

$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$ [Pascal's Identity.]

S



S is a set of size $n+1$

Proof: A is the set of k-element subsets of S.

$$\binom{n+1}{k} \cancel{= |A|} = \binom{n}{k} + \binom{n}{k-1}$$

A_1 = the sets in A that don't include x: $\binom{n}{k} = |A_1|$.

A_2 = the sets in A that do include x. $\binom{n}{k-1} = |A_2|$.

~~|A|~~ $|A| = |A_1 \cup A_2| = |A_1| + |A_2| = \binom{n}{k} + \binom{n}{k-1}$

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          1
        1   1
      1   2   1
    1   3   3   1
  1   4   6   4   1
1   5  10  10   5   1
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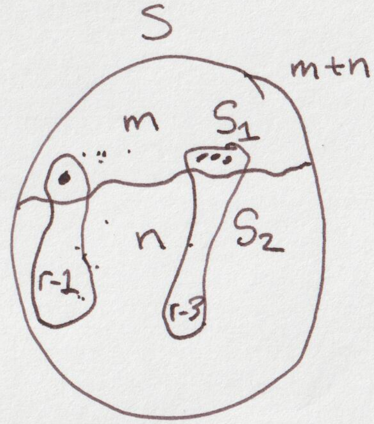
Theorem: $\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$ [Vandermonde's Identity].



Proof: Let S be a set of size $m+n$

Let A be the set of r-element subsets of S.

$$\binom{m+n}{r} \cancel{= |A|} = \sum \binom{m}{k}\binom{n}{r-k}$$

For each $k \in \{0, \dots, \cancel{k}\}$. Let A_k be the sets in A that contain exactly k elements from S_1

$$|A| = |A_0 \cup A_1 \cup \cdots \cup A_r|$$

$$= |A_0| + |A_1| + \cdots + |A_r| = \binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \cdots \binom{m}{r}\binom{n}{0}$$

$$|A_k| = \binom{m}{k} \cdot \binom{n}{r-k} \qquad = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}.$$