Recursively Defined Functions

n	f(n).
0	3
1	9.
2	21
3	45
4	⋮

$$f(n) = \begin{cases} 3 & \text{if } n=0 \\ 2\cdot f(n-1)+3 & \text{if } n \geq 1.. \end{cases}$$

$f(1) = 2\cdot f(0)+3 = 2\cdot 3+3 = 9.$

$f(2) = 2\cdot f(1)+3 = 2\cdot 9+3 = 21$

$f(3) = 2\cdot f(2)+3 = 2\cdot 21+3 = 45$

Claim: $f(n) = 3\cdot 2^{n+1} - 3.$ for all integers $n \geq 0.$

$3\cdot 2^{0+1} - 3 = 3\cdot 2 - 3 = 6-3 = 3 = f(0)$. ✓

$3\cdot 2^{1+1} - 3 = 9 = f(1)$. ✓

$3\cdot 2^{2+1} - 3 = 21 = f(2)$ ✓

Proof by induction on n:

Base case $n=0$ $\quad 3 = f(n) = 3 = 3\cdot 2^{0+1} - 3$ ✓

Assume that $f(k) = 3\cdot 2^{k+1} - 3$ for all $k \in \{0,1,2,\ldots, n-1\}$.
and all $n \geq 1$

For $n \geq 1$,

$f(n) = 2\cdot f(n-1) + 3$ [def'n of f(n)].

$= 2(3\cdot 2^{n-1+1} - 3) + 3$

$= 2(3\cdot 2^{n} - 3) + 3$

$= 3\cdot 2^{n+1} - 6 + 3$

$= 3\cdot 2^{n+1} - 3.$ ✓ QED.

$$g(n) = \begin{cases} 1 & \text{if } n=0 \\ n \cdot g(n-1) & \text{if } n \geq 1. \end{cases}$$

$$\begin{aligned} g(n) &= n \cdot g(n-1) \\ &= n \cdot (n-1) \cdot g(n-2) \\ &= n(n-1) \cdot (n-2) \cdot g(n-3) \\ &\vdots \\ & n(n-1)(n-2) \cdots\cdots 3 \cdot 2 \cdot 1 \cdot \overset{\overset{1}{\shortparallel}}{g(0)} \\ &= n! \end{aligned}$$

$$B(n,k) = \begin{cases} 1 & \text{if } k=0 \text{ and } n \geq 0. \\ 1 & \text{if } k=n \geq 0 \\ B(n-1,k) + B(n-1,k-1) & \text{if } k \in \{\underbrace{1,2,\ldots,n-1}\} \text{ and } n \geq 2. \end{cases}$$

Pascal's Identity

$$B(n,k) = \binom{n}{k}.$$

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

$$f_n = \begin{cases} 0 & \text{if } n=0 \\ 1 & \text{if } n=1 \\ f_{n-1}+f_{n-2} & \text{if } n \geq 2. \end{cases}$$

"Fibonacci" Function

n	f_n
0	0
1	1
2	1
3	2
4	3
5	5
6	8
7	13
8	21

$ax^2+bx+c=0.$

$$\underset{a}{1}\quad \underset{b}{-1}\quad \underset{c}{-1}$$

$x^2-x-1=0.$

$f_1 = f_0 + f_{-1}$ ← problem

$f_2 = f_1 + f_0.$

b

solutions to $x^2 = x+1$

__Theorem:__ For every integer $n \geq 0$,

$$f_n = \frac{\varphi^n - \psi^n}{\sqrt{5}}, \text{ where } \varphi = \frac{1+\sqrt{5}}{2} \text{ and } \psi = \frac{1-\sqrt{5}}{2}$$

>1 b $<1.$

"phi"   "psi"

"golden ratio"   "ugly little brother"

Proof: Base cases $n=0$ $\quad \dfrac{\varphi^0-\psi^0}{\sqrt{5}} = \dfrac{1-1}{\sqrt{5}} = 0 = f_0 = 0.$ ✓

$n=1$ $\quad \dfrac{\varphi^1-\psi^1}{\sqrt{5}} = \dfrac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}{\sqrt{5}.} = \dfrac{\frac{\sqrt{5}}{2}+\frac{\sqrt{5}}{2}}{\sqrt{5}} = \dfrac{\sqrt{5}}{\sqrt{5}} = 1 = f_1$ ✓

Assume that $f_k = \dfrac{\varphi^k - \psi^k}{\sqrt{5}}$ for all $k \in \{0, \dots, n-1\}$.

$$f_n = f_{n-1} + f_{n-2}$$

$$= \frac{\varphi^{n-1} - \psi^{n-1}}{\sqrt{5}} + \frac{\varphi^{n-2} - \psi^{n-2}}{\sqrt{5}}$$

$$= \frac{\varphi^{n-1} - \psi^{n-1} + \varphi^{n-2} - \psi^{n-2}}{\sqrt{5}} = \frac{\varphi^{n-1} + \varphi^{n-2} - \psi^{n-1} - \psi^{n-2}}{\sqrt{5}}$$

$$= \frac{\varphi^{n-2}(\varphi + 1) - \psi^{n-2}(\psi + 1)}{\sqrt{5}}$$

$$= \frac{\varphi^{n-2} \cdot \varphi^2 - \psi^{n-2} \cdot \psi^2}{\sqrt{5}} = \frac{\varphi^n - \psi^n}{\sqrt{5}}.$$

Magic!

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S_n = the set of all n-bit strings that don't contain 00

$B_n = |S_n|$

$$\rightarrow S_0 = \{\varepsilon\} \qquad B_0 = 1$$

$$\rightarrow S_1 = \{0, 1\}. \qquad B_1 = 2$$

$$S_2 = \{\cancel{00}, 01, 10, 11\} \qquad B_2 = 3$$

$$S_3 = \left\{\begin{matrix}\cancel{000}, \cancel{001}, 010, 011 \\ \cancel{100}, 101, 110, 111\end{matrix}\right\} \qquad B_3 = 5.$$

$$B_n = \begin{cases} 1 & \text{if } n=0 \\ 2 & \text{if } n=1 \\ B_{n-1} + B_{n-2}. & \text{if } n \geq 2. \end{cases}$$

$S_{n,1}$ = strings in S_n that start with 1.

$S_{n,0}$ = strings in S_n that start with 0.

$$\boxed{1}\underbrace{\text{anything from } S_{n-1}}_{n-1}$$

$$\boxed{0}\boxed{1}\underbrace{\text{anything from } S_{n-2}}_{n-2}.$$

$$B_n = |S_n| = |S_{n,1} \cup S_{n,0}| = |S_{n,1}| + |S_{n,0}|$$

$$= B_{n-1} + B_{n-2}.$$

$$B_n = B_{n-1} + B_{n-2}$$

n	B_n	
0	1	
1	2	$= f_3$
2	3	
3	5	
4	8	
5	13.	

$$B_n = f_{n+2}$$

Q_n = the strings of length n over the alphabet $\{a,b,c\}$. that don't contain aa

$\cancel{b}$ ab

$Q_0 = \{\varepsilon\}$.

$Q_1 = \{a, b, c\}$.

$Q_2 = \{\text{everything except } aa\}$

$$|Q_n| = \begin{cases} 1 & \text{if } n=0 \\ 3 & \text{if } n=1. \\ 2|Q_{n-1}| + 2\cdot|Q_{n-2}| & \text{if } n \geq 2. \end{cases}$$

$\boxed{b}\ \underbrace{\text{anything from } Q_{n-1}}_{n-1}$ .. $Q_{n,b}$

$\boxed{c}\ \underbrace{\text{anything from } Q_{n-1}}$ .. $Q_{n,c}$

2

$\boxed{a}\boxed{b/c}\ \underbrace{\text{anything from } Q_{n-2}}$. $Q_{n,a}$

$ab\ \underbrace{Q_{n-2}}$

$ac\ \underbrace{Q_{n-2}}$