Infinite Probability Spaces (S, Pr)

Non-empty countable set

$$Pr : S \to [0,1], \quad \sum_{\omega \in S} Pr(\omega) = 1$$

Example: Toss a coin until the first time it comes up head.

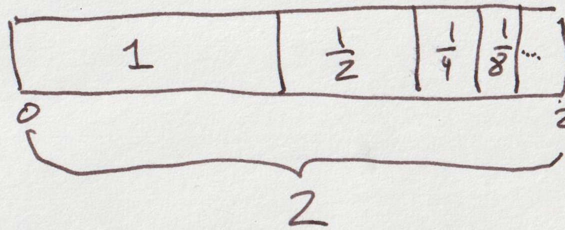
$$S = \{H, TH, TTH, TTTH, TTTTH, \ldots\ldots\}$$

$$= \{T^n H : n \geq 0\} \qquad Pr(T^n H) = \left(\frac{1}{2}\right)^{n+1}$$

$$\sum_{\omega \in S} Pr(\omega) = \sum_{n=0}^{\infty} Pr(T^n H) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} = \frac{1}{2} \overbrace{\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n}}^{2} = \frac{1}{2} \cdot 2 = 1 \qquad \stackrel{?}{=} 1$$

$$\lim_{N \to \infty} \sum_{n=0}^{N} \left(\frac{1}{2}\right)^n = \lim_{N \to \infty} \frac{1 - \left(\frac{1}{2}\right)^{N+1}}{1/2} = \lim_{N \to \infty} \left(2 - \left(\frac{1}{2}\right)^N\right) = 2 - 0 = 2.$$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots\cdots$$



Claim: $\displaystyle\sum_{n=0}^{N} x^n = \frac{1 - x^{N+1}}{1 - x}$, for any $x \neq 1$.

$$(1-x)(x^0 + x^1 + x^2 + x^3 + \cdots + x^N) \stackrel{?}{=} \frac{1 - x^{N+1}}{\cancel{1-x}}$$

$$\begin{aligned} &x^0 + \cancel{x^1} + \cancel{x^2} + \cancel{x^3} + \cdots\cdots + \cancel{x^N} \\ &\quad - \cancel{x^1} - \cancel{x^2} - \cancel{x^3} + \cdots\cdots - \cancel{x^N} - x^{N+1} = 1 - x^{N+1}. \end{aligned}$$

Claim: $\displaystyle\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, for any $-1 < x < 1$

Example: First head wins.

- P_1 and P_2 take turns tossing a coin.
- The first player to toss a H wins.

$S = \{T^n H : n \geq 0\}$. $Pr(T^n H) = \left(\frac{1}{2}\right)^{n+1}$.

$$E = \text{"}P_1 \text{ wins this game"} = \{\overset{0}{H}, \overset{2}{\overbrace{TT}}H, \overset{4}{\overbrace{TTTT}}H \ldots\} = \{T^{2K} H : K \geq 0\}$$

$\frac{1}{2}$ $\frac{1}{8}$

$$Pr(E) = \sum_{\omega \in E} Pr(\omega) = \sum_{K=0}^{\infty} Pr(T^{2K} H) = \sum_{K=0}^{\infty} \left(\frac{1}{2}\right)^{2K+1} = \frac{1}{2} \sum_{K=0}^{\infty} \left(\frac{1}{2}\right)^{2K} = \frac{1}{2} \sum_{K=0}^{\infty} \left(\frac{1}{4}\right)^{K}.$$

$$= \frac{1}{2} \cdot \frac{1}{1 - 1/4} = \frac{1}{2} \cdot \frac{1}{3/4} = \frac{1}{2} \cdot \frac{4}{3} = \frac{2}{3}.$$

$$a^{b \cdot c} = (a^b)^c$$

$$\left(\frac{1}{2}\right)^{2K} = \left(\left(\frac{1}{2}\right)^2\right)^K$$

$$= \left(\frac{1}{4}\right)^K.$$

Example: Second head wins.

$S=\{T^nHT^mH : n\geq 0, m\geq 0\}$. $\quad Pr(T^nHT^mH)=\left(\frac{1}{2}\right)^{n+m+2}$

$A=$ "Player 1 wins this game"

$=\{T^nHT^mH : n+m \text{ is odd}\}$.

$A_1=$ "P_1 tosses the first head and the second head" $=\{T^{2k}HT^{2l+1}H : k,l\geq 0\}$.

$A_2=$ "P_2 tosses the first head and P_1 tosses the second head" $=\{T^{2k+1}HT^{2l}H : k,l\geq 0\}$.

$$Pr(A_1)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}Pr(T^{2k}HT^{2l+1}H)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left(\frac{1}{2}\right)^{2k+2l+3}$$

$$=\frac{1}{8}\sum_{k=0}^{\infty}\left(\sum_{l=0}^{\infty}\left(\frac{1}{2}\right)^{2k+2l}\right)=\frac{1}{8}\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^{2k}\sum_{l=0}^{\infty}\left(\frac{1}{2}\right)^{2l}$$

$$=\frac{1}{8}\cdot\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^{2k}\cdot\frac{4}{3}=\frac{1}{6}\cdot\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^{2k}=\frac{1}{6}\cdot\frac{4}{3}=\frac{2}{9}.$$

$Pr(A_2)=\frac{2}{9}$.

$$Pr(A)=Pr(A_1\cup A_2)=Pr(A_1)+Pr(A_2).$$
$$=\frac{2}{9}+Pr(A_2).$$
$$=\frac{2}{9}+\frac{2}{9}$$
$$=\frac{4}{9}.$$

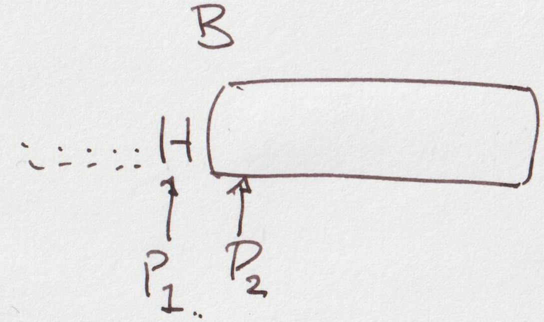
Law of Total Probability.

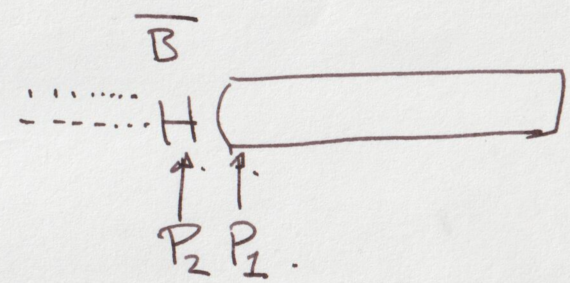
$$\Pr(A) = \Pr(B)\cdot\Pr(A\mid B) + \Pr(\bar{B})\cdot\Pr(A\mid\bar{B}).$$

$$\cancel{\Pr(B)}\cdot\frac{\Pr(A\cap B)}{\cancel{\Pr(B)}} + \cancel{\Pr(\bar{B})}\cdot\frac{\Pr(A\cap\bar{B})}{\cancel{\Pr(\bar{B})}}.$$

B = "P_1 wins Game 1"

A = "P_1 wins Game 2"





$$\Pr(A) = \Pr(B)\cdot\Pr(A\mid B) + \Pr(\bar{B})\cdot\Pr(A\mid\bar{B}).$$

$$= \frac{2}{3}\cdot\underbrace{\Pr(A\mid B)} + \frac{1}{3}\cdot\Pr(A\mid\bar{B})$$

$$= \frac{2}{3}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{2}{3} = \frac{2}{9} + \frac{2}{9} = \frac{4}{9}.$$