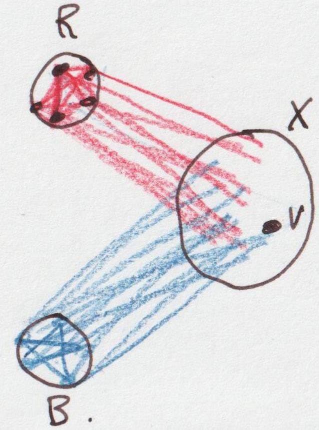
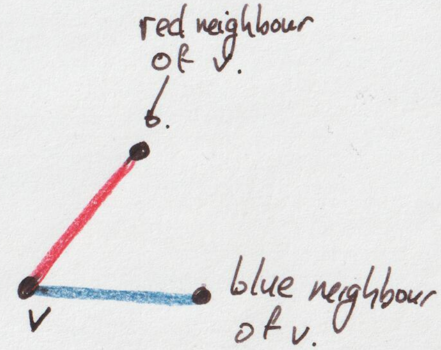
<u>Theorem</u>: Let G be an n-vertex complete graph whose edges are coloured red and blue. Then G contains a monochromatic clique of size at least $\frac{1}{2}(\log_2(n+1) - 1)$.



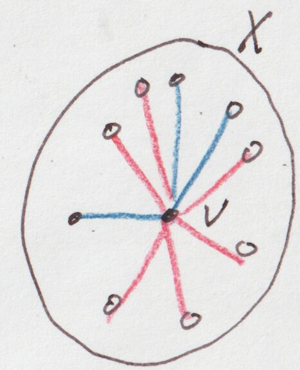
<u>Plan</u>: Keep 3 sets of vertices, R, B, X.

- For every $v \in R$, every neighbour of v in $R \cup X$ is a red neighbour.
- For every $v \in B$, every neighbour of v in $B \cup X$ is a blue neighbour.



$R = \emptyset$, $B = \emptyset$, $X = V(G)$.

While X is not empty,
  let v be any vertex in X.
  if v has at least $(|X|-1)/2$ red neighbours in X.
    $R = R \cup \{v\}$.
    $X = X \setminus (\{v\} \cup \{\text{every blue neighbour of } v \text{ in } X\})$.
  else
    $B = B \cup \{v\}$.
    $X = X \setminus (\{u\} \cup \{\text{every red neighbour of } v \text{ in } X\})$.



Let n_i = the size of X after i steps.

$n_0 = n.$, $\quad n_1 \geq (n_0 - 1)/2 = \frac{n_0}{2} - \frac{1}{2}.$

$n_2 \geq (n_1 - 1)/2 \geq \left(\frac{n_0}{2} - \frac{1}{2} - 1\right)/2 = \frac{n_0}{4} - \frac{1}{4} - \frac{1}{2}$

$R(K,K) \approx 4^K$

$n_3 \geq \frac{n_0}{8} - \frac{1}{8} - \frac{1}{4} - \frac{1}{2}$

$n_i \geq \frac{n}{2^i} - \left(1 - \frac{1}{2^i}\right)$

$\frac{n}{2^i} > 1 - \frac{1}{2^i} \iff \frac{n + 1}{2^i} > 1.$ $\quad R(K,K) \leq (4-\varepsilon)^K$

bigger than 0.

$n+1 > 2^i$

$\Updownarrow$

$\log_2(n+1) > i.$

This process runs for at least $\lfloor \log_2(n+1) \rfloor$ step.

$(3.999975)^K.$

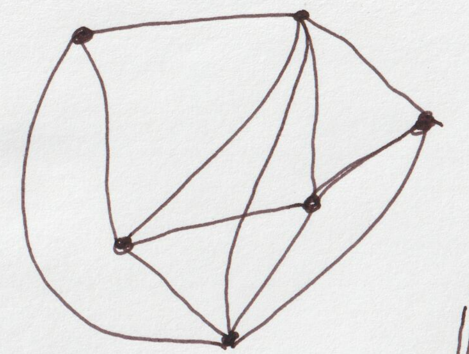
$\lfloor \log(n+1) \rfloor \geq \log(n+1) - 1$

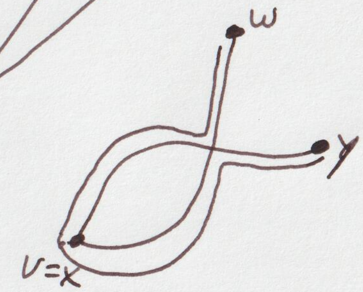
$|R| + |B| \geq \log(n+1) - 1 \qquad \max\{|R|, |B|\} \geq \frac{1}{2}\left(\log_2(n+1) - 1\right).$ QED.

Crossing Lemma.

- A crossing is between two edges xy and vw with no endpoints in common, $\{x,y\} \cap \{v,w\} = \emptyset$

Embedding:

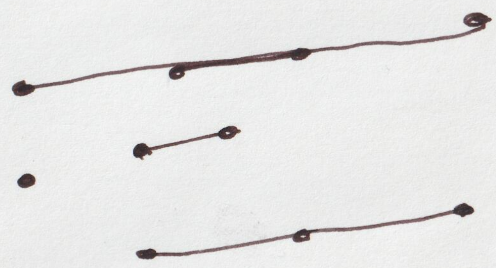




Planar graph: A graph that has an embedding without crossings.

Lemma: Any planar graph with ~~n vertices~~ $n \geq 3$ vertices has at most $3n-6$ edges.

Lemma: Any planar graph with n vertices has at most $3n$ edges.

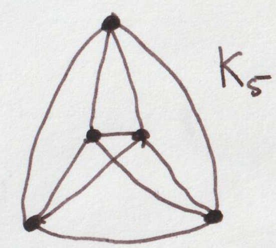


Definition: For any graph G, $cr(G)$ is the minimum number of crossings in any embedding of G.

Lemma: If G has n vertices and m edges then $cr(G) \geq m - 3n$.

Proof: Induction on m.

Base case $m = 3n$, $cr(G) \geq 0$ is trivial.



$1 \leq cr(K_5) \leq 1$

$cr(K_5) = 1$.

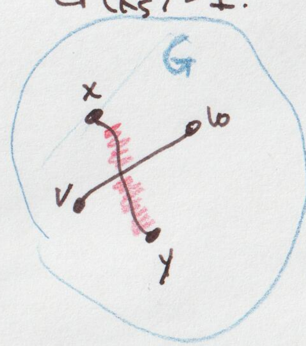
$n=5$.

$\binom{5}{2} = 10$ edges

$3 \cdot 5 - 6 = 9$ edges.

I.H. Assume for any graph H with $k<m$ edges and n vertices, $cr(H) \geq k - 3n$.

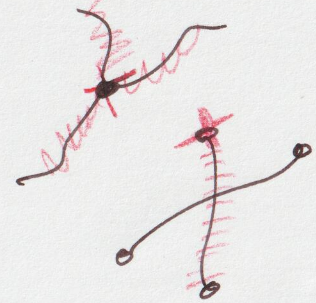
Prove statement for $m > 3n$. ~~Since~~ Consider an embedding of G with $cr(G)$ crossings. Since $m > 3n$, this embedding has a crossing pair of edges xy and vw. Let $H = G - xy$.



$$m-1-3n \leq cr(H) \leq cr(G)-1$$

$$\boxed{cr(K_n) \geq \binom{n}{2} - 3n = \frac{n^2}{2} - O(n)}$$

$$cr(K_n) \simeq C \cdot \frac{n^6}{n^2} = C \cdot n^4.$$



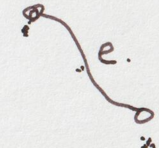
Crossing Lemma: For any n-vertex graph G with $m \geq 4n$ edges,

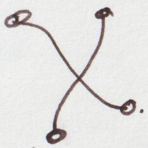
$$cr(G) \geq \frac{1}{64}\left(\frac{m^3}{n^2}\right)$$

Proof: Consider some drawing of G with $cr(G)$ crossings.

Fix some $p \in [0,1]$. For each vertex v of G, Keep v with probability p and remove v with prob. $1-p$. Call the resulting graph G_p.

- n_p = #vertices in G_p.
- m_p = #edges in G_p.
- X_p = #crossing in this embedding of G_p.





$$X_p \geq m_p - 3n_p \qquad E(X_p) \geq E(m_p - 3n_p) = E(m_p) - 3 \cdot E(n_p).$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$p^4 \cdot cr(G) \geq m p^2 - 3 p \cdot n \qquad \text{Let } p = \frac{4n}{m} \leq 1$$

$$cr(G) \geq \frac{m}{p^2} - \frac{3n}{p^3} = \frac{m}{(4n/m)^2} - \frac{3n}{(4n/m)^3} = \frac{m^3}{16n^2} - \frac{3m^3}{64n^2}$$

$$= \frac{4m^3}{64n^2} - \frac{3m^3}{64n^2}$$

$$\frac{m^3}{64n^2}!$$