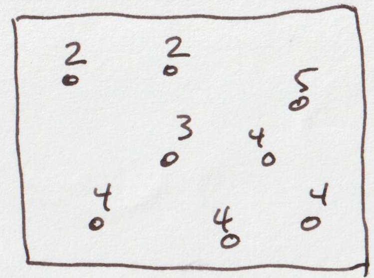
Probability Space (S, Pr).

S



Def'n: A <u>random variable</u> is any function $X : S \to \mathbb{R}$ [?!?]

S ↑ outcomes, $\mathbb{R}$ ↑ real number.

Example: Roll 2 dice

- $S = \{(i,j) : i,j \in \{1,2,3,4,5,6\}\}$ $\quad Pr(\omega) = \frac{1}{|S|} = \frac{1}{36}$ for any $\omega \in S$.

- $X((i,j)) = i+j$

Example: Toss a coin 3 times.

$S = \{TTT, TTH, THT, THH, HTT, HTH, HHT, HHH\}$. $\quad Pr(\omega) = \frac{1}{|S|} = \frac{1}{8}$ for all $\omega \in S$.

$X(\omega) =$ "the number of H in ω"

$$X(\omega) = \begin{cases} 0 & \text{if } \omega = TTT \\ 1 & \text{if } \omega \in \{HTT, THT, TTH\}. \\ 2 & \text{if } \omega \in \{HHT, HTH, THH\}. \\ 3 & \text{if } \omega = HHH \end{cases}$$

$$Y(\omega) = \begin{cases} 1 & \text{if } \omega \in \{HHH, TTT\}. \\ 0 & \text{otherwise.} \end{cases}$$

"$X = r$" $= \{\omega \in S : X(\omega) = r\}$.

X ↑ r.v. $\quad r$ ↑ real number

Random variables define events.

- "$X = 3$" $= \{HHH\}$.
- "$X = 2$" $= \{HHT, HTH, THH\}$.

$Pr(X=0) = 1/8$.

$Pr(X=1) = 3/8$

$Pr(X=2) = 3/8$.

$Pr(X=3) = Pr(\{HHH\}) = \frac{1}{8}$.

$Pr(Y=1) = \frac{2}{8} = \frac{1}{4}$.

$Pr(Y=0) = \frac{6}{8} = \frac{3}{4}$.

<u>Definition</u>: Random variables X and Y are independent if for every $x, y \in \mathbb{R}$.

$$Pr(X=x \text{ and } Y=y) = Pr(X=x) \cdot Pr(Y=y).$$

[ie. the events "$X=x$" and "$Y=y$" are independent].

$Pr(X=2 \text{ and } Y=1) \neq Pr(X=2) \cdot Pr(Y=1)$.

$= Pr(\{HHT, HTH, THH\} \cap \{HHH, TTT\})$

$= Pr(\emptyset) = 0$.

$= \frac{3}{8} \cdot \frac{1}{4} \neq 0$

$\therefore$ X and Y are not independent random variables.

$$Z(\omega) = \begin{cases} 1 & \text{if } \omega \in \{HTT, HTH, HHT, HHH\}. \\ 0 & \text{if } \omega \in \{TTT, TTH, THT, THH\} \end{cases}$$

Are Z and Y independent?

$Pr(Z=1) = \frac{4}{8} = \frac{1}{2}$ $Pr(Z=0) = \frac{4}{8} = \frac{1}{2}$.

$Pr(Z=0 \text{ and } Y=0) = Pr(\{TTH, THT, THH\}) = \frac{3}{8} = \frac{1}{2} \cdot \frac{3}{4} = Pr(Z=0) \cdot Pr(Y=0)$. ✓

$Pr(Z=0 \text{ and } Y=1) = Pr(\{TTT\}) = \frac{1}{8} = \frac{1}{2} \cdot \frac{1}{4} = Pr(Z=0) \cdot P(Y=1)$. ✓

$Pr(Z=1 \text{ and } Y=0) = Pr(\{HTT, HTH, HHT\}) = \frac{3}{8} = Pr(Z=1) \cdot Pr(Y=0)$. ✓

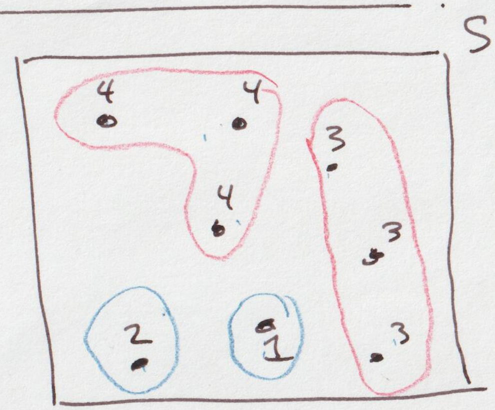
$Pr(Z=1 \text{ and } Y=1) = Pr(\{HHH\}) = \frac{1}{8} = \frac{1}{2} \cdot \frac{1}{4} = Pr(Z=1) \cdot Pr(Y=1)$ ✓

Def'n: Random variables $X_1 \ldots X_n$ are $\left\langle \begin{matrix}\text{pairwise} \\ \text{mutually}\end{matrix} \right\rangle$ independent if for every $x_1, x_2, \ldots, x_n \in \mathbb{R}$ the events "$X_1 = x_1$", "$X_2 = x_2$", ..., "$X_n = x_n$" are $\left\langle \begin{matrix}\text{pairwise} \\ \text{mutually}\end{matrix} \right\rangle$ independent.

---

expectation/mean.

Def'n: The expected value of a random variable X is



$$E(X) = \underbrace{\sum_{\omega \in S} \Pr(\omega) \cdot X(\omega)}_{|S| \text{ terms.}} = \underbrace{\sum_{x} \Pr(X = x) \cdot x}_{|\text{Range}(X)| \text{ terms.}}$$

Example: Toss a coin $S = \{H, T\}$ $\Pr(H) = \Pr(T) = \frac{1}{2}$

$$X(\omega) = \begin{cases} 1 & \text{if } \omega = H \\ 0 & \text{if } \omega = T \end{cases}$$

$$\begin{aligned} E(X) &= \Pr(H) \cdot X(H) + \Pr(T) \cdot X(T). \\ &= \tfrac{1}{2} \cdot 1 + \tfrac{1}{2} \cdot 0 \\ &= \tfrac{1}{2}. \end{aligned}$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Example: Roll a 6-sided die.

$S = \{1, 2, 3, 4, 5, 6\}$. $\Pr(\omega) = \frac{1}{6}$ for each $\omega \in S$.

$X(i) = i$ $\qquad E(X) = \sum_{i=1}^{6} \Pr(i) \cdot i = \sum_{i=1}^{6} \frac{1}{6} \cdot i = \frac{1}{6} \cdot \sum_{i=1}^{6} i = \frac{1}{6} \cdot \frac{6 \cdot 7}{2} = \frac{7}{2} = 3.5.$

$$Y(i) = \frac{1}{i} \qquad E(Y) = \sum_{i=1}^{6} Pr(i) \cdot Y(i) = \sum_{i=1}^{6} \frac{1}{6} \cdot \frac{1}{i} = \frac{1}{6} \sum_{i=1}^{6} \frac{1}{i} = \frac{49}{120}.$$

$S = \{(i,j) : i,j \in \{1,2,3,4,5,6\}\}$. $\quad Pr(\omega) = \frac{1}{36}$ for each $\omega \in S$.

$X((i,j)) = i+j$. $\quad [\text{Range}(X) = \{2,3,4,5,\dots,12\}.]$

$$E(X) = \sum_{x=2}^{12} Pr(X=x) \cdot x = Pr(X=2) \cdot 2 + Pr(X=3) \cdot 3 + Pr(X=4) \cdot 4 + \dots + Pr(X=12) \cdot 12.$$

$$= \frac{1}{36} \cdot 2 + \frac{2}{36} \cdot 3 + \frac{3}{36} \cdot 4 + \dots + \frac{6}{36} \cdot 7 + \frac{5}{36} \cdot 8 + \dots + \frac{1}{36} \cdot 12$$

$$= 7.$$

Linearity of Expectation: For any two random variables X and Y

$$E(X+Y) = E(X) + E(Y).$$

$X+Y$ is the random variable Z, defined by $Z(\omega) = X(\omega) + Y(\omega)$.

Proof: $$E(X+Y) = \sum_{\omega \in S} (X(\omega) + Y(\omega)) \cdot Pr(\omega).$$

$$= \underbrace{\sum_{\omega \in S} Pr(\omega) \cdot X(\omega)}_{E(X)} + \underbrace{\sum_{\omega \in S} Pr(\omega) \cdot Y(\omega)}_{E(Y)}.$$