$\binom{n}{k}$ = # of k-element subsets of an n-element set.

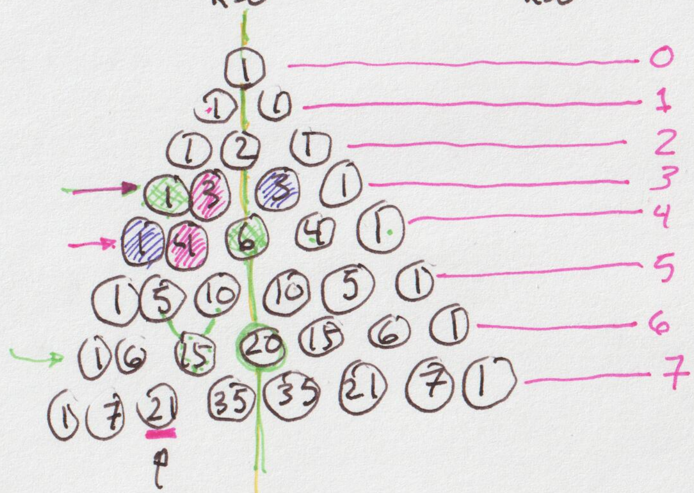
Pascal: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$

Vandermonde: $\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$

→ Easy: $\binom{n}{k} = \binom{n}{n-k}$

$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$

$$\left[\text{Proof: } \binom{2n}{n} = \binom{n+n}{n} = \sum_{k=0}^{n} \binom{n}{k}\underbrace{\binom{n}{n-k}}_{=\binom{n}{k}} = \sum_{k=0}^{n} \binom{n}{k}^2\right]$$



$1^2+3^2+3^2+1^2$

$=1+9+9+1=20$

$$\binom{7}{2} = \binom{3+4}{2} = \binom{3}{0}\binom{4}{2} \quad 1\cdot 6$$

$$+\binom{3}{1}\binom{4}{1} \quad 3\cdot 4$$

$$+\binom{3}{2}\binom{4}{0} \quad 3\cdot 1$$

$$=21$$

Q: How many rearrangements are there of the word "PIGEONS"

1 2 3 4 5 6 7

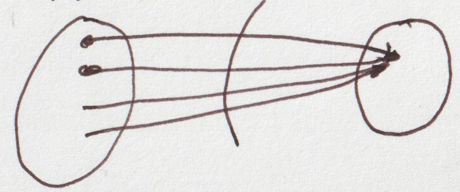
A: $7!$

Q: How many rearrangements of the word "SUCCESS"

1 2 3 4 5 6 7

SUEC SSC

SUEC CSS



C S U C S _ S

1 2 3 4 5 6 7

S x 3	1. Choose 3 locations for S	$\binom{7}{3}$
U x 1	2. Choose 1 location for U	$\binom{4}{1}$
C x 2	3. Choose 2 locations for C	$\binom{3}{2}$
E x 1	4 Place E	$\binom{1}{1}$

A: $\binom{7}{3}\binom{4}{1}\binom{3}{2}\binom{1}{1} =$

A: $\dfrac{7!}{3!\,2!}$

$\parallel$

A: $\binom{7}{2}\binom{5}{3}\binom{2}{1}\binom{1}{1} = \dfrac{7\cdot 6}{2!}\cdot\dfrac{5\cdot 4\cdot 3}{3!}\cdot\dfrac{2}{1}\cdot 1 = \dfrac{7!}{2!\,3!}$

$\binom{7}{2} = \dfrac{7!}{2!\,5!}$

$\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$

Q: How many non-negative integer solutions are there to $x_1+x_2+x_3=11$?

S = "set of solutions" S_i = solutions where $x_1=i$

$x_1=0$

$x_2+x_3=11-i$

x_2

$A_1 = |S| = |S_0 \cup S_1 \cup S_2 \cup \cdots \cup S_{11}|$

$= |S_0| + |S_1| + \cdots + |S_{11}|$

$= \sum_{i=0}^{11} (11-i+1)$

$\boxed{\text{Theorem 3.9.1 Useful.}}$ 3.9.2

$= 11-i$

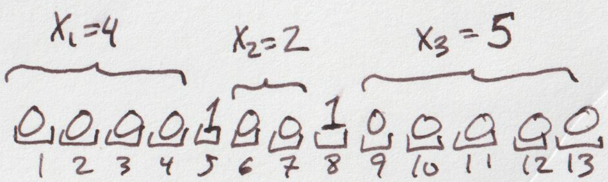
$\to x_2+x_3 = 5$

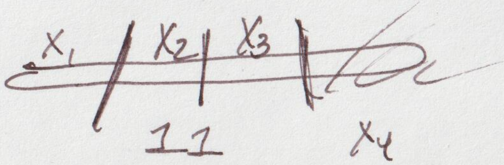
$|S_i| = 11-i+1$

$A_2 = \binom{13}{2}$

Step 1: choose x_2 (6 options).

Step 2: $x_3 = 5 - x_2$ (1 options)





$x_2+x_3=5$.

	x_3
$x_2 = 0$	5
1	4
2	3
3	2
4	1
5	0

Theorem (3.9.1). The number of non-negative integer solutions to $x_1 + x_2 + x_3 + \cdots + x_k = n$ is

$$\binom{n+k-1}{k-1}$$

Theorem: 3.9.2 The number of non-negative integer solutions to $x_1 + x_2 + \cdots + x_k \leq n$ is

$$\binom{n+k}{k}$$

Q: How many solutions to $x_1 + x_2 + x_3 \leq 11$?

non-negative integer

A: Sum Rule: S_i = solutions to $x_1 + x_2 + x_3 = i$.

$$|S_0 \cup S_1 \cup S_2 \cup \cdots \cup S_{11}|$$

$$= \sum_{k=0}^{11} \binom{k+2}{2}$$

A_2: Use theorem 3.9.1 with $k=4$ $n=11$.

$$\binom{14}{3}$$

Solutions to $x_1 + x_2 + x_3 + x_4 = 11$

$x_1 + x_2 + x_3 \leq 11$

$$x_4 = 11 - (x_1 + x_2 + x_3)$$

Bijection between solutions to

$$x_1 + x_2 + x_3 + x_4 = 11$$

and solutions to

$$x_1 + x_2 + x_3 \leq 11$$