

COMP2804A Midterm Exam

Fall 2019

- All questions must be answered on the scantron sheet.
- Write your name and student number on the scantron sheet.
- You do not have to hand in this examination paper.
- Calculators are allowed.

Marking Scheme: Each of the 17 questions is worth 1 mark.

- Binomial coefficients:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

- Newton's Binomial Theorem:

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$

- Fibonacci numbers:

$$f_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ f_{n-1} + f_{n-2} & \text{if } n \geq 2 \end{cases}$$

1. Let $n \geq 8$ be an even integer and let $S = \{1, 2, 3, \dots, n\}$. Consider 7-element subsets of S that consist of 4 even numbers and 3 odd numbers. How many such subsets are there?

(a) $\binom{n/2}{4} \cdot \binom{n/2}{3}$

(b) $\binom{n}{4} \cdot \binom{n}{3}$

(c) $\binom{n/2}{4} + \binom{n/2}{3}$

(d) $\binom{n}{4} + \binom{n}{3}$

2. Let $s \geq 1$, $t \geq 1$, and $k \geq 1$ be integers. The Carleton Computer Science Society is organizing their annual Halloween party. At this party, there are

- s students who are dressed up as Superman,
- t students who are dressed up as Donald Trump,
- k students who are dressed up as Kim Jong Un.

These $s + t + k$ students are arranged on a line, such that all Supermen are standing next to each other, all Trumps are standing next to each other, all Kims are standing next to each other, and no Trump is standing next to any Kim. How many such arrangements are there?

(a) $(s + t + k)! / (s! \cdot t! \cdot k!)$

(b) $(s + t + k)! / (2 \cdot s! \cdot t! \cdot k!)$

(c) $s! \cdot t! \cdot k!$

(d) $2 \cdot s! \cdot t! \cdot k!$

3. Let $n \geq 1$ be an integer. Consider functions $f : \{1, 2, 3, \dots, n\} \rightarrow \{1, 2, 3, \dots, 7n\}$ such that, for each $i \in \{1, \dots, n\}$, $f(i)$ is divisible by 7 (in other words, $f(i)/7$ is an integer). How many such functions are there?

(a) $(7n)^n$

(b) n^{7n}

(c) n^n

(d) 7^n

4. Let $m \geq 5$ and $n \geq 5$ be integers. Let M be a set consisting of m strictly positive integers, and let N be a set consisting of n strictly negative integers. Consider 5-element subsets A of $M \cup N$ such that the product of the elements in A is strictly positive. How many such subsets A are there?

- (a) $\binom{n}{5} + \binom{n}{3} \cdot \binom{m}{2} + n \cdot \binom{m}{4}$
- (b) $\binom{m}{5} + \binom{m}{3} \cdot \binom{n}{2} + m \cdot \binom{n}{4}$
- (c) $\binom{m+n}{5} - \binom{n}{5}$
- (d) $\binom{m}{5} \cdot \binom{n}{5}$

5. Let $n \geq 2$ be an even integer and let $S = \{1, 2, 3, \dots, n\}$. Consider subsets of S that contain at least one even number. How many such subsets are there?

- (a) $(n/2) \cdot 2^{n/2}$
- (b) $2^{n/2} + 2^{n/2}$
- (c) $2^n - 2^{n/2}$
- (d) $2^n + 2^{n/2}$

6. Let $n \geq 7$ be an integer and consider strings of length n over the alphabet $\{a, b, c, d\}$. How many such strings are there that start with abc or end with $dddd$?

- (a) $4^{n-3} + 4^{n-4}$
- (b) $4^{n-3} + 4^{n-4} - 4^{n-7}$
- (c) $2 \cdot 4^{n-3} - 4^{n-7}$
- (d) $2 \cdot 4^{n-4} - 4^{n-7}$

7. Let $n \geq 2$ be an integer. What does

$$\sum_{k=2}^n \binom{n}{k} \cdot 2^{n-k}$$

count?

- (a) The number of strings of length n , over the alphabet $\{a, b\}$, that contain at least one a .
- (b) The number of strings of length n , over the alphabet $\{a, b\}$, that contain at least 2 many as .
- (c) The number of strings of length n , over the alphabet $\{a, b, c\}$, that contain at least one a .
- (d) The number of strings of length n , over the alphabet $\{a, b, c\}$, that contain at least 2 many as .

8. Consider a square with sides of length 17. This square contains n points. What is the minimum value of n such that we can guarantee that at least two of these points have distance at most $17/\sqrt{2}$?
- (a) 4
 (b) 5
 (c) 6
 (d) 7
9. What is the coefficient of $x^{20}y^{80}$ in the expansion of $(7x - 13y)^{100}$?
- (a) $\binom{100}{80} \cdot 7^{20} \cdot 13^{80}$
 (b) $-\binom{100}{80} \cdot 7^{20} \cdot 13^{80}$
 (c) $\binom{100}{20} \cdot 7^{80} \cdot 13^{20}$
 (d) $-\binom{100}{20} \cdot 7^{80} \cdot 13^{20}$
10. A bitstring is called 01-free if it does not contain 01. Let $n \geq 2$ be an integer. How many 01-free bitstrings of length n are there?
- (a) $n - 1$
 (b) n
 (c) $n + 1$
 (d) $n + 2$
11. A bitstring is called 00-free if it does not contain two 0s next to each other. In class, we have seen that for any $m \geq 1$, the number of 00-free bitstrings of length m is equal to the $(m + 2)$ -th Fibonacci number f_{m+2} .
- What is the number of 00-free bitstrings of length 20 that have 0 at position 7? (The positions are numbered $1, 2, \dots, 20$.)
- (a) $f_7 \cdot f_{15}$
 (b) $f_8 \cdot f_{14}$
 (c) $f_8 \cdot f_{15}$
 (d) $f_7 \cdot f_{14}$

12. The functions $f : \mathbb{Z}_{\geq 0}^2 \rightarrow \mathbb{Z}_{\geq 0}$ and $g : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$ are recursively defined as follows:

$$f(m, n) = \begin{cases} m & \text{if } n = 0 \\ 1 + f(m, n - 1) & \text{if } n \geq 1 \end{cases}$$

$$g(n) = \begin{cases} 1 & \text{if } n = 0 \\ 5 \cdot g(n - 1) & \text{if } n \geq 1 \end{cases}$$

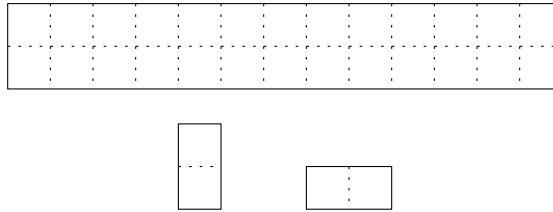
For any integer $n \geq 0$, what is $f(g(n), g(n))$?

- (a) $2 \cdot 5^n$
 - (b) $2 \cdot 5^{n-1}$
 - (c) $(5^n)^2$
 - (d) $(5^{n-1})^2$
13. Consider strings of characters over the alphabet $\{a, b, c\}$. Such a string is called *ccc-free* if it does not contain *ccc*.

For any integer $n \geq 0$, let B_n be the number of *ccc-free* bitstrings of length n . Which of the following is true, for every integer $n \geq 4$?

- (a) $B_n = B_{n-1} + B_{n-2} + B_{n-3}$
- (b) $B_n = 2 \cdot B_{n-1} + 2 \cdot B_{n-2} + 2 \cdot B_{n-3}$
- (c) $B_n = 2 \cdot B_{n-1} + 2 \cdot B_{n-2} + B_{n-3}$
- (d) $B_n = B_{n-1} + B_{n-2} + 2 \cdot B_{n-3}$

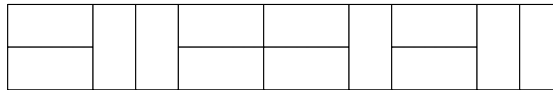
14. Let $n \geq 1$ be an integer and consider a $2 \times n$ board B_n consisting of a total of $2n$ square cells. The top part of the figure below shows B_{13} .



A *brick* is a horizontal or vertical board consisting of 2 square cells; see the bottom part of the figure above. A *tiling* of the board B_n is a placement of bricks on the board such that

- the bricks exactly cover B_n and
- no two bricks overlap.

The figure below shows a tiling of B_{13} .



Let T_n be the number of different tilings of the board B_n . Which of the following is true for any $n \geq 3$?

- $T_n = T_{n+1} + T_{n+2}$
- $T_n = T_{n-1} + T_{n-2}$
- $T_n = 2 \cdot T_{n-1} + T_{n-2}$
- $T_n = T_{n-1} + 2 \cdot T_{n-2}$

15. Consider the recursive algorithm HELLO, which takes as input an integer $n \geq 0$:

HELLO(n) :

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if  $n = 0$  or  $n = 1$  then
  print "hello"
else
  HELLO( $n - 1$ )
  print "hello"
  HELLO( $n - 2$ )

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If you run algorithm HELLO(5), how many times is the word "hello" printed?

- 13
- 14
- 15
- 16

16. Let $X = \{1, 2, \dots, 20\}$. You choose a uniformly random 7-element subset Y of X . Define the event $A = \text{"3 is an element of } Y \text{ or } 13 \text{ is an element of } Y\text{"}$. What is $\Pr(A)$?

(a) $\frac{\binom{19}{6} + \binom{19}{6}}{\binom{20}{7}}$

(b) $\frac{2 \cdot \binom{20}{6} - \binom{20}{5}}{\binom{20}{7}}$

(c) $\frac{2 \cdot \binom{19}{7} - \binom{18}{7}}{\binom{20}{7}}$

(d) $\frac{2 \cdot \binom{19}{6} - \binom{18}{5}}{\binom{20}{7}}$

17. After having proctored the midterm, Alexa, Farah, May, and Shelly decide to go trick-or-treating. For any house that the ladies visit, if they do not receive candy, they throw rotten eggs at the house.

Let $m \geq 7$ and $n \geq 7$ be integers. There are m houses whose owners hand out candy and there are n houses whose owners do not hand out candy.

The ladies choose a uniformly random subset of 7 houses and visit these 7 houses. Define the event $A = \text{"the ladies throw rotten eggs at exactly 2 of the 7 chosen houses"}$. What is $\Pr(A)$?

(a) $\frac{\binom{m}{5} + \binom{n}{2}}{\binom{m+n}{7}}$

(b) $\frac{\binom{m}{5} \cdot \binom{n}{2}}{\binom{m+n}{7}}$

(c) $\frac{\binom{m+n}{7}}{\binom{m}{5} \cdot \binom{n}{2}}$

(d) $\frac{\binom{7}{2}}{\binom{m+n}{7}}$

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