# COMP2804 Midterm Exam, Fall 2024, Section B

This a closed book exam with a duration of 1h20m. No computers, calculators, phones, or other aids are allowed.

This paper has wide margins. Feel free to write in them. You **may not** take this paper with you when you are done.

This is a multiple-choice Scantron exam. Be sure to complete your name and student number and answer all questions on the Scantron sheet provided to you.

Select exactly one option for each question. In cases where you believe there is more than one correct option, select the most accurate option.

**Marking scheme:** Each of the 17 questions is worth 1 mark.

**Reminders:**

- Binomial coefficients:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

- Newton's Binomial Theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

- Fibonacci numbers:

$$f_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ f_{n-1} + f_{n-2} & \text{if } n \geq 2 \end{cases}$$

**Question 1.** Pat's Pub has just introduced the *Super Value Meal.* When you order this meal you get to choose one drink (Coke, Sprite, Fanta, or Water), one main (hamburger, cheeseburger, nachos, or chicken wings), two different sides (two of: fries, onion rings, rice, vegetables, cheese sticks), and one dessert (chocolate cake, key lime pie, ice cream, or fruit salad).

How many meal options does a customer at Pat's Pub have?

(a) $4 + 4 + 5 + 4$
(b) $4 \cdot 4 \cdot 5 \cdot 4$
(c) $4 \cdot 4 \cdot 5^2 \cdot 4$
(d) $4 \cdot 4 \cdot \binom{5}{2} \cdot 4$
(e) $4 \cdot 4 \cdot 2^5 \cdot 4$

**Question 2.** The WhizBang Computer Company builds custom gaming PCs. Customers can choose one of five different cases, one of four different CPUs, one of six different graphics cards, and *up to* two different add-ons: LEDs, super-bright LEDs, red LEDs, blue LEDs, or a gaming mouse.

How many options does a WhizBang customer have when buying a custom gaming PC?

(a) $5 + 4 + 6 + 5$
(b) $5 \cdot 4 \cdot 6 \cdot 2 \cdot 5$
(c) $5 \cdot 4 \cdot 6 \cdot \binom{5}{2}$
(d) $5 \cdot 4 \cdot 6 \cdot (5 + \binom{5}{2})$
(e) $5 \cdot 4 \cdot 6 \cdot (6 + \binom{5}{2})$

**Question 3.** The school of Computer Science has $n$ professors:

- $g$ of the professors self-identify as *green*
- $b$ of the professors self-identify as *blue*
- the remaining $n - g - b$ professors self-identify as *other*.

(Any individual professor is either green, blue, or other, so this partitions the professors into three disjoint sets.) We need to form a committee with 10 members. For EDI purposes, the committee must contain at least one green member and at least one blue member. How many ways are there to form this committee?

(a) $\binom{n}{10}$
(b) $\binom{n}{10} - \binom{g}{10} - \binom{b}{10}$
(c) $\binom{n}{10} - \binom{n-g}{10} - \binom{n-b}{10}$
(d) $\binom{n}{10} - \binom{g}{10} - \binom{b}{10} + \binom{n-g-b}{10}$
(e) $\binom{n}{10} - \binom{n-g}{10} - \binom{n-b}{10} + \binom{n-g-b}{10}$

**Question 4.** I'm having a dinner party with 9 guests (so 10 people, including myself, will be attending). To promote a sense of unity, everyone will be sitting at a long table with 10 chairs all on the same side of the table. Two of my guests, Kamala and Donald, dislike each other so they can't be seated side-by-side. How many ways are there to seat my guests so that Donald is not sitting next to Kamala?

(a) $10!$
(b) $9! \cdot 8$
(c) $8! \cdot 2!$
(d) $8! \cdot \binom{9}{2}$
(e) $10! - 9!$

**Question 5.** Continuing the previous question, two more of my guests (Justin and Catherine) really like each and insist on sitting beside each other.

How many ways are there to seat my guests?

(a) $9!$
(b) $9! \cdot 8$
(c) $7! \cdot 3!$
(d) $7! \cdot \binom{9}{2}$
(e) $2 \cdot (9! - 8!)$

**Question 6.** Nate doesn't have any cats, so he's going to the animal shelter to adopt some. There are $c$ cats at the shelter and Nate will adopt 0, 1, or 2 of them. The person working at the shelter informs Nate that a house with $x$ cats needs $x + 1$ litter boxes which Nate can buy at the shelter. There are 10 different colours of litter boxes. Nate needs to choose $x \in \{0, 1, 2\}$ cats to adopt and buy $x + 1$ litter boxes, for which he needs to choose the colours.

For example, Nate might say "I'll adopt Figaro and Garfield and I'll buy one green and two red litter boxes. No brown litter boxes for me, thank you very much."

Or Nate might say "I don't want to adopt any of these cats, but I guess I have to buy one litter box anyway, make it a blue one, to match my eyes."

How many options does Nate have?

(a) $3 \cdot c \cdot 10^3$
(b) $3 \cdot \binom{c}{3} \cdot \binom{10}{3}$
(c) $10 + c \cdot 10^2 + \binom{c}{2} \cdot 10^3$
(d) $10 + c \cdot \binom{11}{9} + \binom{c}{2} \cdot \binom{12}{9}$
(e) $10 + 10c + 10c^2$

**Question 7.** How many strings can be obtained by rearranging the letters of the word

C H E E R L E S S N E S S

(a) $13!$

(b) $13!/(3 \cdot 2^6)$

(c) $13 \cdot 12 \cdot \binom{11}{4} \cdot 7 \cdot \binom{6}{4}$

(d) $\binom{13}{4} \cdot \binom{9}{4} \cdot 5!$

(e) $\binom{13}{3} \cdot \binom{10}{3} \cdot \binom{7}{3} \cdot \binom{4}{3}$

**Question 8.** Which of the following does $\sum_{k=0}^{n-1} \binom{n}{k} 3^k$ count?

(a) The number of strings over the alphabet $\{a, b, c\}$.
(b) The number of strings over the alphabet $\{a, b, c\}$ that contain at least one $a$.
(c) The number of strings over the alphabet $\{a, b, c, d\}$.
(d) The number of strings over the alphabet $\{a, b, c, d\}$ that contain at least one $a$.
(e) The number of strings over the alphabet $\{a, b, c, d\}$, minus one.

**Question 9.** How many ways are there to partition a set of size $3n$ into three sets, each of size $n$? (Note that a partition is a set of sets, so we consider $\{\{A, B\}, \{C, D\}, \{E, F\}\}$ and $\{\{C, D\}, \{B, A\}, \{E, F\}\}$ to be the same partition of $\{A, B, C, D, E, F\}$. Since Halloween is close, think of putting six different pieces of candy into three identical paper bags.)

(a) $3^{3n}$

(b) $\binom{3n}{n}\binom{2n}{n}\binom{n}{n}$

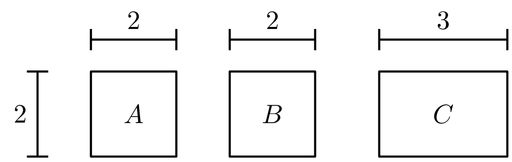
(c) $(3n)!/(6(n!)^3)$

(d) $(3n)^3$

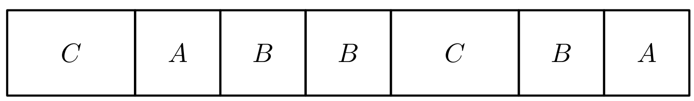
(e) $\binom{3n}{2n}\binom{2n}{n}$

**Question 10.** I have three kinds of tiles

- Alabaster tiles of size $2 \times 2$
- Brown tiles of size $2 \times 2$
- Canary yellow tiles of size $3 \times 2$




For any integer $n \geq 2$, let $F(n)$ be the number of ways these types of tiles can be used to tile a $n \times 2$ rectangle. For example, here is a tiling of the $16 \times 2$ rectangle:




Which of the following is true about $F(n)$?

(a) $F(n) = f_n$, for all $n \geq 0$ (remember than $f_n$ is the $n$th Fibonacci number)
(b) $F(n) = 2 \cdot F(n-2) + F(n-3)$ for all $n \geq 3$
(c) Both of the above
(d) $F(n) = F(n-2) + F(n-3)$ for all $n \geq 3$
(e) $F(n) = f_{n+2}$

**Question 11.** Define the function

$$G(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ 1 & \text{if } n = 2 \\ 2 \cdot G(n-2) + G(n-3) & \text{if } n \geq 3 \end{cases}$$

Which of the following is true about $G(n)$?

(a) $G(n) = f_n$ for all $n \geq 0$
(b) $G(n) = 2^n$
(c) $G(n) = (\frac{5}{2})^n$
(d) $G(n) = (\frac{5}{2})^n - n$
(e) $G(n) = (\frac{5}{2})^n - \sqrt{2n}$

**Question 12.** A bitstring is called 00-free if it does not contain two consecutive zeros. In class, we have seen that, for any integer $m \geq 1$ the number of 00-free bitstrings of length $m$ is equal to the $(m+2)$th Fibonacci number $f_{m+2}$.

What is the number of 00-free bitstrings of length 20 that have a 0 at position 10? More precisely, how many 00-free bitstrings are there of the form $b_1, \ldots, b_9, 0, b_{11}, \ldots, b_{20}$?

(a) $f_9 + f_{10}$
(b) $f_9 \cdot f_{10}$
(c) $f_{10} \cdot f_{11}$
(d) $f_{11} \cdot f_{12}$
(e) None of the other answers is correct

**Question 13.** Let $Q(n)$ be the number of ways that we can place $2n$ students at $n$ distinct (round, chair-free) desks so that exactly two students are sitting at each desk. Since the desks are round, the two students sitting a desk are unordered—each desk is assigned a 2-element subset of the $2n$ students. Which of the following is true about $Q(n)$?

(a) $Q(1) = 1$

(b) $Q(n) = \binom{2n}{2} \cdot Q(n-1)$ for all $n \geq 1$

(c) $Q(n) = (2n)!/2^n$

(d) All of the above

(e) None of the above

**Question 14.** Consider the following (very slow) algorithm for computing the $n$th Fibonacci number $f_n$:

```
FIB(n)
  if n ≤ 1 then
    return n
  return FIB(n − 1) + FIB(n − 2)
```

When running FIB(95), how many times is FIB(90) called?

(a) 2
(b) 3
(c) 5
(d) 8
(e) 13

**Question 15.** I have a collection of 100 MP3 files and I need to make a playlist with $n$ songs. I am allowed to play the same song more than once, but I am not allowed to play the same song three times in a row.

Let $R(n)$ be the number of ways I can make an $n$-song playlist from 100 songs where the playlist never plays the same song three times in a row. Then $R(0) = 1$ since there is only one way to make an empty playlist. Which of the following is true about $R$:

(a) $R(1) = 100$
(b) $R(2) = 10000$
(c) $R(n) = 9900 \cdot R(n-2) + 9900 \cdot R(n-3)$, for all $n \geq 3$.
(d) All of the above
(e) None of the above

**Question 16.** You roll three six-sided dice, one red, one green, and one blue. Let $R$, $G$, and $B$ be the values facing up on the red, blue, and green die, respectively. Define the event:

$$A = \text{“}R = G = B\text{”} \ .$$

Which of the following is true:

(a) $\Pr(A) = 1/36$
(b) $\Pr(A) = 1/18$
(c) $\Pr(A) = 1/12$
(d) $\Pr(A) = 1/9$
(e) $\Pr(A) = 1/6$

**Question 17.** You roll three six-sided dice, one red, one green, and one blue. Let $R$, $G$, and $B$ be the values facing up on the red, blue, and green die, respectively. Define the event:

$$E = \text{“}R + G = 4\text{”} \ .$$

Which of the following is true:

(a) $\Pr(E) = 1/36$
(b) $\Pr(E) = 1/18$
(c) $\Pr(E) = 1/12$
(d) $\Pr(E) = 1/9$
(e) $\Pr(E) = 1/6$